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PRIME AND MAXIMAL IDEALS IN Γ - SEMIGROUPS

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Abstract:

In this paper we introduce and study the structure of prime and maximal ideals of Γ-semigroup. In this paper many important results of prime ideals in semigroups have been extended to prime ideals in Γ-semigroups.

1. Introduction

As a generalization of semigroup, SEN [9,10], introduced the notion of Γ-semigroup in 1986 and developed some theory on Γ-semigroups. In fact the concept of Γ-semigroups is a generalization of the concept of semigroups. Due to the fact it is easy to generalize the result of semigroups to Γ-semigroups. The motivation mainly comes from the conditions of prime ideals and maximal that are importance and interest in semigroups. We can see that any semigroup can be reduced to Γ-semigroup. In this article, we give some auxiliary results are also necessary for our considerations and characterize the prime and maximal ideals in Γ-semigroups. Many classical notions of semigroup have extended to Γ- semigroups[1-8]

Definition 2.1: [9] Let S and Γ be two nonempty sets. Then S is called a Γ- semigroup if there exist a mapping from S×Γ×S to S which maps $(a,\alpha,b) \rightarrow a\alpha b$ satisfying the condition $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a,b,c \in T$ and $\gamma,\mu \in \Gamma$. Let S be a Γ-semigroup. If *A* and *B* are two subsets of S, we shall denote the set $\{a\gamma b : a \in A, b \in B$ and $\gamma \in \Gamma\}$ by $A\Gamma B$.

Example 2.2: Let Q be the set of rational numbers and Γ = N be the set of natural numbers. Define a mapping from Q×Γ×Q to Q by aαb = usual product of a,α,b for $a,b \in Q$, $\alpha \in \Gamma$. Then Q is a Γ-semigroup.

Definition 2.3:A Γ-semigroup S is said to be a commutative provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$. If S is a commutative Γ-semigroup then $a\Gamma b = b\Gamma a$ for all $a,b \in S$.

Definition 2.4: A nonempty subset A of a Γ-semigroup S is said to be a left Γ-ideal of S if $s\in S$, $a\in A$, $\alpha\in\Gamma$ implies $s\alpha a\in A$. A nonempty subset A of a Γ-semigroup S is a left Γ-ideal of S if $S\Gamma A\subseteq A$. A nonempty subset A of a Γ-semigroup S is a right Γ-ideal of S if $A\Gamma S\subseteq A$. A nonempty subset A of a Γ-semigroup S is a Γ-ideal of S if and only if it is both a left Γ-ideal and a right Γ-ideal of S.

Example 2.5: Let N be the set of natural numbers and $\Gamma = 2N$. Then N is a Γ-semigroup and $A = 3N$ is a Γ-ideal of the Γ-semigroup N.

Definition 2.6: A Γ-ideal A of a Γ-semigroup S is said to be a maximal Γ-ideal provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

Definition 2.7:A Γ-ideal A of a Γ-semigroup S is said to be a proper Γ-ideal of S if A is different from S.

Definition 2.8: A proper ideal P of a Γ-semigroup S is called a prime ideal of S if $A\Gamma B\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$ for any two ideals A,B of S.

Theorem 2.9: If M is a maximal ideal of a Γ-semigroup S and if the complement of M is commutative and is contained in S^2, then M is a prime ideal of S.

Proof: The theorem is trivial if $M = S$. Let M be a proper maximal ideal and suppose M is not prime; then there exist an elements x and y in $C(M)$ such that $x\alpha y\in M$; $\alpha\in\Gamma$. Let x be some fixed element of $C(M)$ such that, for atleast one element $y\in C(M)$; $x\alpha y\in M$, $\alpha\in\Gamma$ and let Y be the set of all such elements y. We distinguish two cases: (1) $Y\subset C(M)$. (2) $Y = C(M)$. Case (1): In this case we shall show that $M\cup Y$ is an ideal of S. i.e $(M\cup Y)\Gamma S\subseteq M\cup Y$ and $S\Gamma(M\cup Y)\subseteq M\cup Y$. Since $S = M\cup C(M)$ we may accomplish this by showing that $(M\cup Y)\Gamma M\subseteq M\cup Y$ and $M\Gamma(M\cup Y)\subseteq M\cup Y$; which is obvious since M is an ideal and that $(M\cup Y)\Gamma C(M)\subseteq M\cup Y$ and $C(M)\Gamma(M\cup Y)\subseteq M\cup Y$. Now If $y\in Y$ and $c\in C(M)$ then $x\alpha(y\beta c) = (x\alpha y)\beta c\in M$ for $x\alpha y\in M$, $\alpha, \beta\in\Gamma$. Hence either $y\beta c\in M$ or $y\beta c\in Y$, so that $(M\cup Y)\Gamma C(M)\subseteq M\cup Y$. And since we have assumed $C(M)$ to be commutative, $x\alpha(c\beta y) = x\alpha(y\beta c) = (x\alpha y)\beta c\in M$; $\alpha,\beta\in\Gamma$ whence either $c\beta y\in M$ or $c\beta y\in Y$and therefore $C(M)\Gamma(M\cup Y)\subseteq M\cup Y$. This completes the proof that$M\cup Y$ is an ideal. Since Y is nonempty and $M\cap Y = \phi$, $M\subset M\cup Y$ and $M\cup Y\subset S$. Since in this case we have assumed $Y\subset C(M)$. Therefore $M\cup Y$ is a proper ideal of S, properly containing M, contrary to the hypothesis that M is maximal. Case (2): If $Y = C(M)$ then, by virtue of the hypothesis that $C(M)$ is commutative, $y\alpha x = x\alpha y \in M$ for all $y\in C(M)$ and $\alpha\in\Gamma$, whence $x\Gamma s\subseteq M$ and $s\Gamma x\subseteq M$. In particular $x_2\in M$, whence $x \neq x_2$. But since $C(M)\subseteq S^2$, there exist elements a and b (not necessarily distinct) in S such that $x = a\alpha b$ both a and b lie in $C(M)$

since $x \in C(M)$. Now either $a \neq x$ or $b \neq x$, for if $a = b = x$ then $x = a\alpha b = x_2$; say $a \neq x$. Hence $M \cup X$ is an ideal of S, for $(M \cup X)\Gamma S = M\Gamma S \cup X\Gamma S \subseteq M \cup M \subseteq M \subseteq M \cup X$ and $S\Gamma(M \cup X) = S\Gamma M \cup S\Gamma X \subseteq M \cup M \subseteq M \subseteq M \cup X$. But $M \subset M \cup X$ since $x \in C(M)$ and $M \cup X \subset$ S since $a \in C(M)$ and $a \neq x$. Therefore $M \cup X$ is a proper ideal of S, properly containing M, contrary to the hypothesis that M is maximal. The supposition that M is not prime having led to a contradiction in both cases. We conclude that M is prime. We observe that in the above proof of case (1) no use was made of the hypothesis that $C(M) \subseteq S^2$. Hence by combining the maximality of M, the commutativity of $C(M)$ and the condition defining case (1), we obtain.

Corollary 2.10: In a Γ-semigroup S such that S^2 = S, every maximal ideal whose complement is commutative is prime. In particular, in a Γ-semigroup having either a left or a right identity element every maximal ideal whose complement is commutative prime. In a commutative Γ-semigroup such that S^2= S, every maximal ideal is prime. We now prove a converse of Theorem 2.9, and are able to drop the hypothesis that the complement of ideal be commutative.

Theorem 2.11: If M is a maximal and prime ideal of a Γ-semigroup S, then the complement of M is contained in S^2.

Proof: Suppose and let x be any element of $C(M)$ such that $x \notin S^2$ .Then $x \neq x_2$ and $x_2 \in C(M)$. Since M is prime. Now the set $M \cup x_2 \cup S\Gamma x_2 \cup x_2\Gamma S \cup S\Gamma x_2\Gamma S$ is an ideal of S, for $M\Gamma S \subseteq M$; $S\Gamma M \subseteq M$; $S\Gamma S\Gamma x_2 = S2\Gamma x_2 \subseteq S\Gamma x_2$; $x_2\Gamma S\Gamma S = x_2\Gamma S^2 \subseteq x_2\Gamma S$; $S\Gamma x_2\Gamma S = S\Gamma x_2\Gamma S = S\Gamma x_2\Gamma S$; $S\Gamma S\Gamma x_2\Gamma S = S^2\Gamma x_2\Gamma S \subseteq S\Gamma x_2\Gamma S$ and $S\Gamma x_2\Gamma S\Gamma S = S\Gamma x_2\Gamma S^2 \subseteq S\Gamma x_2\Gamma S$. But $M \subset M \cup x_2 \cup S\Gamma x_2 \cup x_2\Gamma S \cup S\Gamma x_2\Gamma S$. Since $x_2 \notin M$ and $M \cup x_2 \cup S\Gamma x_2 \cup x_2\Gamma S \cup S\Gamma x_2\Gamma S \subset S$. Since $x \notin M$ and $x \notin S^2$. Therefore $M \cup x_2 \cup S\Gamma x_2 \cup x_2\Gamma S \cup S\Gamma x_2\Gamma S$ is a proper ideal of S properly containing M, contrary to the hypothesis that M is maximal. We note in passing that if A is any left or right ideal of a Γ-semigroup S and if $C(A) \subseteq S^2$, then $C(A) \subset S^2$ for S^2 is an ideal of S, whence $S^2 \cap A \neq \phi$ while $C(A) \cap A = \phi$ and so $C(A) \neq S^2$. Hence the conclusion of theorem 2.12 would be strengthened to read the complement of M is contained properly in S^2. We note also that, in any Γ-semigroup S, the ideal S^2 is prime if and only if $S^2 = S$and that S^2 is maximal if and only if $C(A) \subset S^2$contains atmost a single element, then only if following from the fact that if $x,y \in CS^2$and $x \neq y$ then $S^2 \cup x$ is a proper ideal of S properly containing S^2. Theorem 2.9 and 2.11 together yield.

Theorem 2.12: In a Γ-semigroup S, a maximal ideal whose complement is commutative is a prime ideal if and only if the compliment is contained in S^2.

Proof: It is well - known theorem 26 [7] in the theory of rings that if *M* is an ideal in a commutative ring *R* then the residue class ring *R/M* is a field if and only if both (1) *M* is a maximal ideal of *R* and (2) $x_2 \in M$ implies $x \in M$. Another familiar result is that an ideal is a commutative ring *R* is a prime ideal of *R* if and only if *R/M* is an integral domain. Since every field is an integral domain (but not conversely), it follows that if conditions (1) and (2) hold then *M* is a prime ideal the converse fails because although every prime ideal satisfies condition (2), a prime ideal need not be maximal. However, it follows that if *M* is a maximal ideal in a commutative ring then *M* is prime if and only if (2) holds and this result we shall prove both for one-sided and for an ideals in Γ-semigroups dispensing with the hypothesis of commutativity. Before proceeding to the proof, we remark that it is easy to show that is a commutative ring with identity element condition (2) follows from condition (1). Whence we conclude that in such a ring every maximal ideal is prime. This result carries over to commutative Γ-semigroup with identity, as we have seen corollary 2.11 indeed, we know that in any Γ-semigroups with identity a maximal ideal whose complement is commutative must be prime.

Conclusion: This concept is used in chemistry, physical chemistry, electronics etc.

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