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ON HYPERGRAPH ASSIGNMENT PROBLEMS

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Abstract

New concepts namely, complete edge-adjacency set and complete edge-disjoint set of a hyperedge with respect to a subset of hyperedge set in a hypergraph are introduced. Based on the newly defined complete sets, a new method namely, non-adjacency search method is proposed for determining all hyperassignments in the given bipartite hypergraph. The solution procedure of the proposed method is presented with an example in Paramedical Technology. Further, the proposed method is extended to find an optimum cost hyperassignment for weighted bipartite hypergraphs.

Keywords: Hypergraph, Hyperassignment, Compete sets, Non-adjacency search method.

1. Introduction

Hypergraph is a generalization of ordinary graph and in it, each edge has any number of vertices. It was widely studied by Berge [1]. Hypergraph is very useful tool to resolve optimization problems such as scheduling problems, location problems, set partitioning problems and so on. In addition, the theory of hypergraphs is seen to be used for solving certain type optimization problems having integer constraints. The classical assignment problem [9] is one of the most important and popular mathematical model in combinatorial optimization. The hyperassignment problem (HAP) [7,8] is a generalization of the classical assignment problem. The HAP has an application in railway vehicle rotation planning [1,2] and flow problems [4,5,6]. An assignment problem in a bipartite hypergraph is termed as a hyper assignment problem. Also, it is called as a set partitioning problem [2]. The HAP on bipartite hypergraphs has been solved using integer linear programming [3].

This paper is structured as follows. Section 2 discusses some basic concepts of the hypergraph and the HAP. In Section 3, two new complete sets in hypergraph namely, complete edge-adjacency set and complete edge-disjoint set of a hyperedge with respect to a subset of hyperedge set in a hypergraph are defined and some of its related results are established. Section 4 proposes a new method namely, non adjacency search method for HAPs. An extended non adjacency search method for solving cost HAP in weighted bipartite hypergraph is presented in Section 5 and finally, the conclusion is given in Section 6.

2. Preliminaries

We need the following definitions related to hypergraphs which can be found in Berge[1] and Borndorfer and Heismann[3].

Definition 2.1 A hypergraph G consists of a pair (X,E) where X is a set of vertices and E is a set of hyperedges where every hyperedge contains any number of vertices from X which is not empty, that is, each hyperedge can be treated as a non-empty subset of X.

Remark 2.1 If each hyperedge of G contains exactly 2 vertices only, then it becomes an ordinary graph.

Definition 2.2 A hypergraph $G=(X,E)$ is said to be bipartite if X can be partitioned into two disjoint subsets X_1 and X_2 of X such that $X=X_1\cup X_2$.

The bipartite hypergraph is represented as $G=(X_1,X_2,E)$ where X_1 and X_2 are two disjoint vertex sets and E is the set of hyperedges. We assume that the partitioned vertex sets have the same cardinality, and that every hyperedge $e\in E$ has the same number $|e\cap X_1|=|e\cap X_2|>0$ of vertices in X_1 and X_2.

Definition 2.3 Let $G=(X_1,X_2,E)$ be a bipartite hypergraph. An assignment in G is a subset E_1 of E such that any two hyperedges in E_1 are nonadjacent .

Definition 2.4. Let $G=(X_1,X_2,E)$ be a bipartite hypergraph. A hyperassignment in G is an assignment H in G such that every $v\in X_1\cup X_2$ is contained in exactly one hyperedge $e\in H$.

3 Complete Sets in Hypergraph

Now, we define two new terms namely, complete edge-adjacency set and complete edge-disjoint set with respect to an edge set $S\subseteq E$ of a hypergraph $G=(V,E)$.

Definition 3.1 Let $G=(V,E)$ where $E=\{e_1,e_2,e_3,...,e_n\}$ be a hypergraph and $S\subseteq E$. A complete edge-adjacency set of a hyperedge $e\in S$ with respect to S is a subset of S, E_1 satisfying the following conditions:

(i) for each $e_i \in E_1$, $e \cap e_i \neq \varnothing$ and (ii) for all $e_k, e_j \in E_1$ with k $\neq$ j, $e_j \cap e_k \neq \varnothing$.

Definition 3.2 Let $G=(V,E)$ where $E=\{e_1,e_2,e_3,...,e_n\}$ be a hypergraph and $S \subseteq E$. A complete edge-disjoint set of a hyperedge $e \in S$ with respect to S is a subset of S, D_1 satisfying the following conditions:

(i) for each $e_i \in D_1$, $e \cap e_i = \varnothing$ and (ii) for all $e_k, e_j \in D_1$ with k $\neq$ j, , $e_j \cap e_k = \varnothing$.

Definition 3.3 Let $G=(V,E)$ where $E=\{e_1,e_2,e_3,...,e_n\}$ be a hypergraph and $S \subseteq E$. S is said to be self complete edge-disjoint if for all $e_k, e_j \in S$ with k $\neq$ j, $e_j \cap e_k = \varnothing$.

Remark 3.1. The definition of self complete edge-disjoint coincides with the definition of hyper matching by Berge [1].

Example 3.1 Consider the hypergraph $G=(V,E)$ where

$V=\{u_1,u_2,u_3,u_4,u_5,u_6,v_1,v_2,v_3,v_4,v_5,v_6\}$, $E=\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$, $e_1=\{u_1,u_2,u_4,v_1,v_2,v_4\}$, $e_2=\{u_1,v_2\}$, $e_3=\{u_2,v_1\}$, $e_4=\{u_2,v_6\}$, $e_5=\{u_3,u_4,v_5,v_6\}$, $e_6=\{u_5,u_6,v_1,v_2\}$ and $e_7=\{u_5,u_6,v_3,v_4\}$.

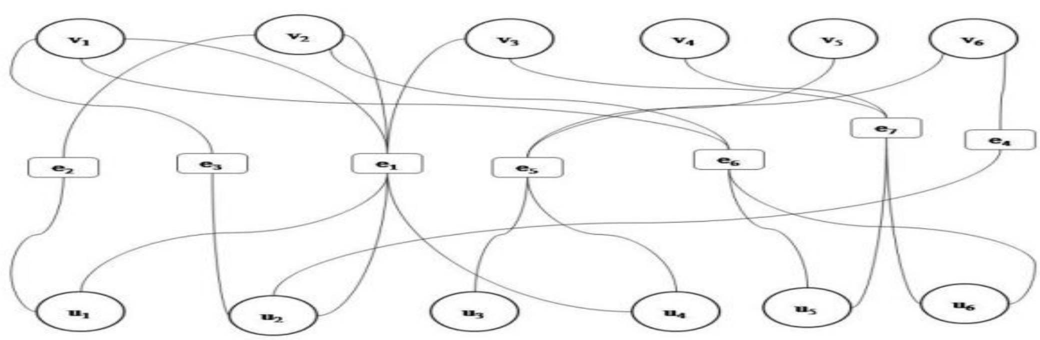

Figure 3.1 Hypergraph $G=(V,E)$.

Now, the complete edge-adjacency set of e_2 with respect to E is $\{e_2, e_1, e_6\}$; the complete edge-disjoint set of e_2 with respect to E is $\{e_3, e_5\}$ and the set $\{e_3, e_5\}$ is a self complete edge- disjoint set .

We, now prove the following theorems which are used in the proposed method namely, non-adjacency search method for finding all hyperassignments in the given hypergraph.

Theorem 3.1 Let G = (U, V, E) be a bipartite hypergraph and $H \subseteq E$. Then, H is an assignment in G if and only if H is a self complete edge-disjoint set.

Proof: Let $H=\{e_1,e_2,e_3,...,e_t\}; 1 \leq t \leq n$.

Assume that H $\subseteq E$ is an assignment in G.

Now, since H is an assignment, we have for all $e_j, e_k \in H$ and $j \neq k$, $e_j \cap e_k = \varnothing$.

This implies that H is a self complete edge-disjoint set.

Assume that $H \subseteq E$ is a self complete edge-disjoint set.

Now, since H is a self complete edge-disjoint set , we have

$e_j \cap e_k = \varnothing$, for all $e_j, e_k \in H$ and $j \neq k$.

This implies that H is an assignment in G.

Hence the theorem is proved.

Theorem 3.2 Let G = (U, V, E) be a bipartite hypergraph. If an assignment $H \subseteq E$ in G with $\bigcup_{e \in H} e = U \cup V$, then H is a hyperassignment in G.

Proof: Now, since $H \subseteq E$ is an assignment in G and by the Theorem 3.1, H is a self complete edge-disjoint set.

Now, since H is a self complete edge-disjoint set and $\bigcup_{e \in H} e = U \cup V$, each vertex in G should be in exactly only one in edge in H. Therefore, H is a hyperassignment in G.

Hence , the theorem is proved.

4. Non-adjacency Search Method

Now, we propose a new method namely, non-adjacency search method to determine all hyperassignments in the given bipartite hypergraph G = (U, V, E) where $E = \{e_1, e_2, ..., e_n\}$.

The proposed method proceeds as:

Step 1: Collect the all non-adjacency edges of each edge in the given hypergraph. Let it be E_i be a set of all non-adjacent edges of e_i, i = 1, 2,..., n .

Step 2: Construct a set $E^* = \{ K \in \{E_1, E_2, E_3, .., E_n\} : \bigcup_{e \in K} e = U \cup V\}$.

Step 3: If $E^* = \varnothing$, then G has no hyperassignment . If not , go to the Step 4..

Step 4: Let $E^* = \{E_1^*, E_2^*, ..., E_k^*\}, 1 \leq k \leq n.$ Take E_1^*.

Step 5: Compute complete edge-adjacency set of each edge of E_1^* with respect to E_1^*.

Step 6: Construct complete edge-disjoint sets with respect to E_1^* using the complete edge-adjacency sets obtained from the Step 5.. Let it be $B=\{B_1, B_2, ..., B_t\}, 1 \le t \le n.$

Step 7: If $B=\varnothing$, then G has no hyperassignment related to E_1^* and then, go to the Step 9..

If not, go to the Step 8..

Step 8: For each i , $1 \le i \le t$, if $\bigcup_{e \in B_i} e = U \cup V$, then B_i is a hyperassignment in G by the Theorem 3.2. and if not, then B_i is an assignment in G by the Theorem 3.1.

Step 9 : Repeat the Step 5. to the Step 8. for $E_2^*, E_3^*, ..., E_k^*$ and then, stop the computation.

Step 10: From the Step 8., all hyper assignments of the given hypergraph are obtained.

The solution procedure of proposed method is illustrated by the following example.

Example 4.1 A health care shop manager wants to buy three different tablets v_1, v_2 and v_3 for three different diseases u_1, u_2 and u_3. The manager knows that two companies e_1 and e_2 produce the tablet v_1 for the disease u_1, two companies e_2 and e_3 produce the tablet v_2 for the disease u_2, one company e_4 produces the tablet v_2 for the disease u_1 and the tablet v_3 for the disease u_3 and one company e_5 produces the tablet v_3 for the disease u_3. The task of the shop manager is to find a set of companies providing tablets for all diseases such that the companies are distinct, every tablet is produced by only one company and every disease is taken care of only one company.

Now, the above said problem can be modeled as a bipartite hypergraph as follows:

The bipartite hypergraph $G=(U_1, V_1, E)$ with $V = U_1 \cup V_1$, $U_1 = \{u_1, u_2, u_3\}$, $V_1 = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, $e_1 = \{u_1, v_1\}$, $e_2 = \{u_1, u_2, v_1, v_2\}$, $e_3 = \{u_2, v_2\}$, $e_4 = \{u_1, u_3, v_2, v_3\}$ and $e_5 = \{u_3, v_3\}$.

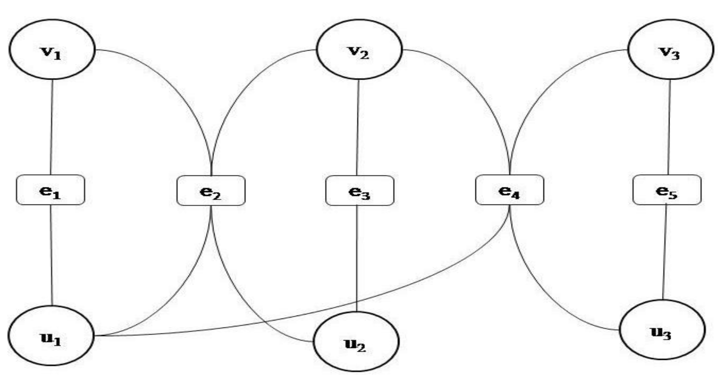

Figure 4.1. Hypergraph $G=(U_1, V_1, E)$

Now, for finding a solution of the task of the manager, we enough to find a hyperassignment in $G=(U_1,V_1,E)$.

Let E_1,E_2,E_3,E_4 and E_5 be the set of all non adjacent edges of e_1,e_2,e_3,e_4 and e_5 respectively in the given hypergraph $G=(U_1,V_1,E)$.

Now, $E_1=\{e_1,e_3,e_5\}$, $E_2=\{e_2,e_5\}$, $E_3=\{e_3,e_1,e_5\}$, $E_4=\{e_4\}$ and $E_5=\{e_1,e_2,e_3,e_5\}$.

Now, since $\bigcup_{e\in E_i} e=V, i=1,2,3,5$ and by the Step 4., each one of the sets E_1, E_2, E_3 and E_5 may give a hyper assignment in $G=(U_1,V_1,E)$.

Now, applying the Step 5. to the Step 8. in $E_1=\{e_1,e_3,e_5\}$, we have that

$\{e_1,e_5\}$ is an assignment in G and $\{e_1,e_3,e_5\}$ is a hyperassignment in G.

Now, applying the Step 5. to the Step 8. in $E_2=\{e_2,e_5\}$, we obtain that $\{e_2,e_5\}$ is a hyperassignment in G.

Now, applying the Step 5. to the Step 8. in $E_5=\{e_1,e_2,e_3,e_5\}$, we get that

$\{e_1,e_3,e_5\}$ and $\{e_2,e_5\}$ are hyperassignments in G.

Therefore, by the Step 10., $H_1=\{e_1,e_3,e_5\}$ and $H_2=\{e_2,e_5\}$ are hyperassignments in G.

Thus, the manager can use any one of the sets $H_1=\{e_1,e_3,e_5\}$ and $H_2=\{e_2,e_5\}$ for purchasing the tablets.

5. Weighted Bipartiate Hypergraphs

In this section, we consider a weighted bipartiate hypergraph and propose an algorithm to find a minimum/ maximum cost hyperassignment for weighted bipartiate hypergraphs.

Let (G, f) be a weighted bipartite hypergraph where G = (U, V, E) is a bipartite hypergraph and a cost function f : E → R , set of real numbers.

Definition 5.1 In (G, f), a hyper assignment H_1 is said to be

(i) a minimum cost hyperassignment in (G, f) if

$f(H_1)$ = min.{ f(H) : H is a hyperassignment in (G, f) } and

(ii) a maximum cost hyperassignment in (G, f) if

$f(H_1)$ = maxi.{ f(H) : H is a hyperassignment in (G, f) }.

Now, a solution procedure based on the non-adjacency search method for finding the minimum/ maximum cost hyperassignment in the weighted bipartite hypergraph (G, f).

The solution procedure proceeds as follows:

Step 1: Determine all hyperassignment in G = (U,V, E) using the non-adjacency search method .

Step 2: If G = (U, V, E) has no hyperassignment, (G, f) has no maximum/ minimum cost hyperassignment.

Step 3: If G = (U, V, E) has a hyperassignment and

(i)If H_1 is a hyperassignment in G with $f(H_1)$ = min.{ $f(H)$: H is a hyperassignment in (G, f)}, H_1 is the minimum cost hyperassignment in (G, f).

(ii)If H_1 is a hyperassignment in G with $f(H_1)$ = maxi.{ $f(H)$:H is a hyperassignment in (G, f)}, H_1 is the maximum cost hyperassignment in (G, f).

Now, the solution procedure for finding minimum and maximum hyperassignments in a weighted bipartite hypergraph is illustrated by the following example.

Example 5.1.

The manager of a health care shop desires to purchase three different tablets v_1, v_2 and v_3 for three different diseases u_1, u_2 and u_3. He knows that two companies e_1 and e_2 produce the tablet v_1 for the disease u_1, three companies e_2 , e_3 and e_4 produce the tablet v_2 for the disease u_2 and two companies e_4 and e_5 produces the tablet v_3 for the disease u_3.

The shop manager gets the profit Rs.4 lakhs, Rs.5 lakhs, Rs.3 lakhs, Rs.7 lakhs and Rs.3 lakhs if he purchases the tablets from the companies e1, e2, e3, e4 and e5 respectively.

The task of the manager is to determine a set of companies providing tablets for all diseases such that the companies are distinct, every tablet is produced by only one company and every disease is taken care of only one company and also, the total profit is maximum.

Now, the above problem can be modeled as a weighted bipartite hypergraph G with cost function f , (G, f) as follows:

$G = (U_1, V_1, E)$, $V = U_1 \cup V_1$, $U_1 = \{u_1, u_2, u_3\}$, $V_1 = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$,

$e_1 = \{u_1, v_1\}$, $e_2 = \{u_1, u_2, v_1, v_2\}$, $e_3 = \{u_2, v_2\}$, $e_4 = \{u_2, u_3, v_2, v_3\}$, $e_5 = \{u_3, v_3\}$, $f(e_1) = 4$, $f(e_2) = 5$, $f(e_3) = 3$, $f(e_4) = 7$ and $f(e_5) = 3$.

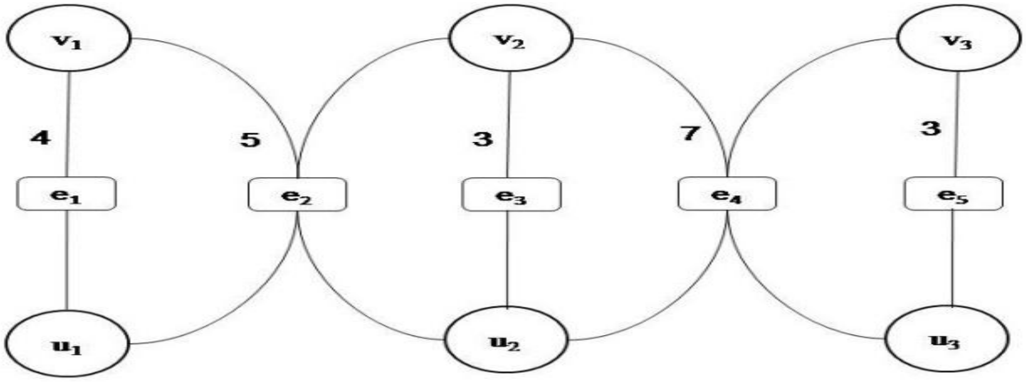

Figure 5.1. Weighted hypergraph (G, f)

Now, for finding a solution of the task of the manager, we enough to find a maximum hyperassignment in (G, f) .

Now, by the Step 1., we have that $\{e_1, e_3, e_5\}, \{e_1, e_4\}$ and $\{e_2, e_5\}$ are hyperassignments in G.

Now, since costs of the hyperassignments $\{e_1, e_3, e_5\}, \{e_1, e_4\}$ and $\{e_2, e_5\}$ are Rs.10 lakhs, Rs.11 lakhs and Rs.8 lakhs respectively and by the Step 3., the maximum cost hyperassignment in (G, f) is $\{e_1, e_4\}$ with total cost Rs.11 lakhs. Thus, if the manager uses the set of companies $\{e_1, e_4\}$ for purchasing the tablets, he gets the maximum total profit Rs. 11 lakhs.

6. Conclusion

In this paper, bipartite hypergraphs with and without cost function are considered. In the hypergraph, new concepts namely, complete edge-adjacency set and complete edge-disjoint set of an hyperedge with respect to a subset of the hyperedge set are defined. A new method namely, non-adjacency search method based on newly defined complete sets is proposed to find all hyperassignments in the bipartite hypergraph and also, to determine a minimum/maximum cost hyperassignment in the weighted bipartite hypergraph. Examples in paramedical technology are used to illustrate the solution procedure of the proposed method. The non-adjacency search method helps decision makers to solve real life problems whose models are hyperassignment problems.

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