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A BRIEF SURVEY ON DOMINATION RELATED PARAMETERS IN GRAPH THEORY

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Abstract

Concurrent to the growth of other research areas of graph theory, the study of *domination in graphs* has encountered a vast growth in the past forty years. Domination plays a vital role in various areas such as communication networks, service locations problems, social network problems, biological model networks etc. Having been motivated by the overwhelming speedy growth and development of the theory of domination in graphs, a brief survey on fundamentals of domination related parameters are presented in this article along with few applications to benefit the readers at a beginning level.

Keywords: graphs, connected graphs, induced subgraphs, domination, vertices, edges

1. Introduction

From the past century, graph theory has found its immense growth in various fields of science and technology. Graph theoretic techniques serve as a convenient tool for the investigation of communication networks. A graph is a mathematical modeling (representation) of any network that depicts the relationship between the objects with some common feature. Henceforth, we use the terms graph and network synonymously.

The origin and development of Graph theory is due to mathematical folklore which subsisted in the form of puzzles. The long standing unsolved puzzle of '*Konigsberg bridge problem*' was solved by the mathematician Leonhard Euler in 1736. He wrote his first paper in graph theory [1] and thus became the father of graph theory. His results lead to the characterization of class of graphs called *Eulerian graphs*. A century later its inception, in 1847, Kirchhoff, developed the theory of *trees* while solving the electrical current equations. The Physicist found that the *spanning trees* efficiently represent the topology of electrical networks and thus can be used in the solutions of simultaneous electrical current equations. Ten years later, Arthur Cayley, a Chemist discovered *trees* independently while investigating the number of structural isomers of the saturated hydrocarbons (paraffin series) $C_k H_{2k+2}$. The concept of a tree is used in so many real life instances such as sorting of mails, binary search methods, networks requiring minimal connectedness, etc. The fun-puzzle game '*Around the World*' invented in 1959, by Sir W.R. Hamilton gave rise to the concept of *Hamiltonian graphs* which in real life context is known as a *Travelling Salesman Problem (TSP)*. No efficient algorithm yet exists to detect whether a given graph is Hamiltonian or not.

Operational researchers have contributed remarkably to the growth of graph theory with the objective of solving practical problems such as shortest distance between the nodes, minimal spanning tree in a graph, maximal network flow in a graph, etc. The *maximal flow problem* proposed and solved by Ford and Fulkerson in 1956 plays a vital role in any type of transportation network.

The concept of *planarity* initiated by Euler was later nurtured by Kuratowski in 1930. Kuratowski characterized planar graphs by a theorem he wrote. The notion of planarity has found several applications in electrical and road networks. In 1936, the psychologist Lewin proposed that the '*life space'* of an individual can be represented by means of a planar graph. The regions of a planar graph resemble an atlas. The *four colour conjecture* according to which 'four colours are sufficient to colour any atlas' was proposed by Mobius in 1840, while attempting to colour the map of the countries of England and still remains a conjecture. The study of colouring related problems gave rise to a wide range of research related topics such as chromatic number, chromatic polynomial, covering, matching and partitioning.

The concept of partitioning of vertices or edges of a graph into sets according to certain properties is applicable to various practical problems such as coding theory, logic partition in digital computers, reduction of states of sequential machines, etc. Some problems of such kind are chromatic partitioning, domination, matching and labeling.

Any interested reader can refer to Harary [2] or any other similar sources to know more details of the discussions that lay above. In [3], Narsingh Deo has exclusively treated the concepts of graph theory from engineering and computer science points of view.

2. Domination Theory – Origin and Growth

The study of *domination in graphs* has found rapid growth in the recent years. It is a highly flourishing area of research in graph theory. So far, hundreds of research articles have appeared on this topic of research in view of its growing real life applications. Let us explain this topic in a precise way. Networks such as an organizational network, traffic (road) network, communication network sometimes require a set of nodes which can control the rest. Stated in other words, in view of reduction of resources and manpower, it is often required to find a set of controlling nodes of smallest size. In graph theoretical terminology, this problem is equivalent to finding a smallest subset D of the vertex set V of a graph G such that every vertex $v \in V$ is either present in D or has a neighbour in D. We say that D *dominates* the vertices of G and is known as a *dominating set* of G. The size of a smallest dominating set is referred to as the *domination number* of the graph G denoted by $\gamma(G)$. The game of '*Chess*' invented by Indians during the medieval period of the first millennium AD was the key force behind the origination of theory of domination in graphs. In 1850's fanciers of chess from Europe pondered over the *Queens problem* of determining the minimum number of queens required to attack every square of the chess board. It was rightly believed to be five. According to the literature survey, Jaenisch was the first researcher who studied the Queens problem mathematically in his article. In 1982, W.W.Rouse Ball described three types of Queen's problems which are related to domination, independent domination and N-queens problem (queens attack all the squares but not themselves). In 1964, the two brothers Yaglom and Yaglom studied these problems in depth and they produced fruitful results for the rook, knight, king and Bishop chess pieces. The book written by Claude Berge [4] in 1958 dealt with the concept of domination for the first time in which he used the term '*coefficient of external stability*' to refer the domination number of a graph. In 1962, another book on graph theory was published by Oystein Ore [5] in which he studied the domination concept using for the first time the terms '*dominating set*' and '*domination number*' with notation '$d(G)$'. In 1977, Cockayne and Hedetniemi published a survey paper on few results available about domination. They were the first to use the symbol $\gamma(G)$ for the domination number of a graph G. This research work is exhorted by the tremendous growth and its underlying practical applications of the field of study. Any interested reader can go through the book written by Haynes et al., [6] or any other related sources to know more on the growth and development of the concept of domination in graphs. The same authors have authored another two more books [7] and [8] on the same issue with advanced topics.

3. Variety of Domination Parameters

Domination is plenty in its variety. In [6], authors cite as much as 75 and more varieties of domination parameters and their work includes the compendium of various domination parameters. Each type of domination meets a specific purpose in real life applications. Various types of dominating sets can be obtained by imposing an additional condition on the dominating set or on the method of dominating. In order to systemize various domination parameters, Harary and Haynes [9] defined the conditional domination number $\gamma(G:P)$ as the minimum cardinality of a dominating set $D \subseteq V$ such that the induced subgraph $\langle D \rangle$ satisfies the property P. Some possible properties imposed on D are:

P1: $\langle D \rangle$ has no edges.

This type of domination is called as *independent domination* and the independent domination number $i(G)$ is the minimum cardinality of an independent dominating set We note that in this case D is an independent set Allan and Laskar [10] and many others have studied this parameter in detail.

P2: $\langle D \rangle$ has no isolated vertices.

This type of dominating set is called as *total dominating set* and its number $\gamma_t(G)$ is the minimum cardinality of such a set Cockayne et al., introduced this parameter in [11].

P3: $\langle D \rangle$ is connected.

This type of domination is called as *connected domination* in graphs and was introduced by Sampathkumar and Walikar in [12]. Its number $\gamma_c(G)$ is defined in a similar manner.

P4: $\langle D \rangle$ is complete.

The dominating set corresponding to this property is called as a *dominating clique* whose number is $\gamma_{cl}(G)$. Cozzens and Kelleher introduced a dominating clique and studied its properties in [13].

P5 $\langle D \rangle$ has a perfect matching.

Haynes and Slater introduced this type of domination in [14] and called such a dominating set as a *paired-dominating set* and denoted its number by $\gamma_p(G)$.

P6 $\langle D \rangle$ is a set of independent edges.

A dominating set with this property is called an *induced-paired dominating set* and its number is $\gamma_{ip}(G)$. This concept was introduced by Studer et al., in [15].

We can still list several such domination parameters. Clearly, $\gamma(G) \le \gamma(G:P)$ for any property P. One can notice that not all conditional domination parameters exist for all graphs. For example, if G has isolated vertices then property P_2 cannot hold. Similarly, if G is disconnected, property P_3 cannot hold.

Analogous to the treatment of inherent properties of a dominating set D the complementary dominating set $V-D$ can also be treated according to the property it inherits. Some of the dominating sets D in this perspective are:

a. *Split dominating set* – The induced subgraph $\langle V-D \rangle$ is disconnected. Its number is denoted by $\gamma_s(G)$.

b. *Nonsplit dominating set* – The induced subgraph $\langle V-D \rangle$ is connected. Its number symbolized as $\gamma_{ns}(G)$.

c. *strong split dominating set*- The induced subgraph $\langle V-D \rangle$ is a null graph. Its number is denoted by $\gamma_{ss}(G)$.

d. *Co-total dominating set* – The induced subgraph $\langle V-D \rangle$ has no isolated vertices. The symbol $\gamma_{\cot}(G)$ stands for its number.

The first three types are introduced by Kulli and Janakiram in [16], [17] and [18] respectively and the last one was introduced by Kulli et al., in [19]. The co-total domination was also independently introduced by Domke et al., in [20] as *restrained domination in graphs*. Kulli and his team of researchers have developed more domination parameters by combining the properties listed above or imposing the conditions of a dominating set D to its complementary set $V-D$.

Apart from these types of domination parameters there are several other types of domination parameters available in literature. Let us glance at some of them once briefly.

1. A dominating set D is a *global dominating set* if it is a dominating set to both the graph G and its complement graph $\overline{G}$. This concept was independently introduced by Brigham and Dutton in [21] where they referred it by the name *factor dominating set* and also by Sampathkumar in [22]. Its number is denoted by $\gamma_g(G)$.

2. The concept of *multiple domination in graphs* was introduced by Fink and Jacobson in [23]. A dominating set in D of a graph G is said to be a $k-$dominating set , if every vertex in $V-D$ is dominated by at least k vertices of D. i.e. $|N(v) \cap D| \ge k$. Its number is $\gamma_k(G)$.

3. The vertices and edges of a graph G are treated as the elements of G. Any set X of elements of G is an *entire dominating set* if every element not in X is either adjacent or incident to at least one element in X. This domination parameter was introduced and studied by Kulli et al., in [24].

4. The set-dominating set of a graph was introduced and studied by Sampathkumar and Pushpalatha in [25]. If the notion of domination between vertices is replaced by domination between sets then it is called as set-domination in graphs. More formally, a dominating set $D \subseteq V$ is called a *set-dominating (sd) set* if for every subset $T \subseteq V - D$, there exists some subset $S \subseteq D$ such that the subgraph $\langle S \cup T \rangle$ induced by the vertices of $S \cup T$ is connected.

5. Another domination parameter viz., point-set domination was developed by the same authors in [26]. A set-dominating set becomes a *point-set dominating (psd) set* if $S = \{v\}$ in the above definition of a set-dominating set.

Comprehensive treatment of variety of domination parameters in graphs can be found in the book written by Haynes et al., [6]. The Indian eminent graph theorist Kulli has surveyed some important results on several domination parameters in his book [27] which serves the readers for a quick review of results on many domination parameters.

The notion of domination is not only limited to the vertices of the graph. It also pertains to the edges of the graph and sometimes even to the mixture of these two items (as in the case of an entire dominating set). Almost all the vertex related domination parameters can be equivalently extended to edge related domination parameters.

A set of edges $X \subseteq E$ is said to be an *edge dominating set* if every edge not in X is adjacent to some edge in X. The edge domination number of a graph G is the cardinality of a smallest edge dominating set of G and is denoted by $\gamma'(G)$. The Edge or line domination was first studied by Gupta in 1969 as line-line covering with the symbol $\alpha_{11}(G)$ for the line-line covering number in [28]. (Similarly he called the domination number as point-point covering number with the symbol $\alpha_{00}(G)$). In 1977, Mitchell and Hedetniemi studied the parameter for trees [29] as edge domination. Arumugam, and Velammal in 1998 [30] studied various parameters of the notion such as connected, total, and independent edge dominating sets.

Recently, in 2008 and in subsequent years, Gayathri and Mohanaselvi contributed few more domination related edge analogues to this field of study. Their conditional edge domination parameters are focused on the properties of the co-edge dominating set $E \setminus X$. Co-edge tree/spanning tree domination [31], co-edge maximal domination [32]. co-edge split/non-split domination, co-edge complete/independent/total domination are their note worthy contributions. These parameters are treated exclusively in [33].

Domination parameters are in general discussed and characterized for their properties and bounds on various graphs. They are also studied for their NP-completeness nature. Their applicative nature in various fields is also studied. Besides these, there is another branch of study viz., the domatic partition or D-partition of graphs introduced by Cockayne and Hedetniemi in [34].

A domatic or D-partition of a graph is a partition of its vertices into dominating sets. The maximum order of such a partition is known as the domatic number of the graph. This notion is of course, extended to other domination parameters with appropriate tags such as connected, independent, total, etc., domatic numbers of a graph and have been studied extensively in the literature.

4. Applications of Domination Parameters

Domination in graphs has a wide range of real life applications. In fact, there exists perhaps no domination parameter without any real life application. Almost all articles and books on domination briefly or elaborately discuss the application nature of the domination parameter under study. We briefly list few examples of such applications from the literature.

The key application scenario of domination problems arises from facility location tasks. Domination is widely used in all branches of telecommunications. Berge discusses the application of domination for the location of radar stations for surveillance purpose in [35]. Liu [36] discusses its use in mounting the transmission stations for communication networks. In any kind of facility locations such as schools, police stations, post offices, fire stations, etc. the use of domination is unavoidable.

Application of domination in coding theory is discussed by Kalbeisch et al., [37] and Cockayne and Hedetniemi [38]. According to their study, if one models the *n*-dimensional vector codes as graphs of hypercubes (*n*-cubes) or any such analogous graphs then the (n, p)-covering sets, perfect covering sets and single error correcting codes are all dominating sets of the graph with some additional properties.

Haynes et al., have distinguished the secondary RNA (Ribonucleic Acid) motifs from other particles in biological models using domination parameters [38]. Milenkovic et al., [64] have explored that a set of protein cells from a minimum dominating set of a protein-protein interaction (PPI) network is topologically central in human PPI network. They find that dominating sets conquer Biologically Central (BC) genes, signaling pathways (SP) genes and drug targets.

An wireless communication system without fixed infrastructures is known as an ad hoc network. Jie Wu and Hailan Li [39], Das and Bhargavan [40] and several others have found that connected dominating sets resemble the virtual backbone of the ad hoc networks used for routing of messages.

A social network theory is concerned about the study of relationships between the members of a group. A social network clique is a dominating clique [13] which is a group of representatives of the network who can communicate themselves directly. A status in a network is a subset S of members of the group such that any two of them have the same relationship outside S in the network. A group of people is said to be structurally equivalent if any two of them have same relationship between people in the social network. Kelleher and Cozzens [41] exhibited that these sets can be determined using the properties of dominating sets.

The notion of domination is used in so many real life instances. In order to minimize the number of locations for a surveyor from which surveying measurements can be made domination is used in land surveying. In an organizational set up domination is used to select a set of representatives from its employees. In computer communication networks domination is used to identify the minimum number of processors required to collect the information from other processors. In the selection of optimal route form, for a bus it is used.

Any reader can further discover more applications on existing domination parameters or invent new domination parameters for some problems in real life context.

5. Conclusion

The huge applicative nature of the concept of domination gave rise to the overwhelming nature of the field of research. In this article, some interesting key note points on background knowledge lying behind graph theory and one of its branches viz., domination are highlighted. The notion of domination is dealt with in detail covering its origin and growth, varieties and applications. The readers would have acquired the basic knowledge about domination related parameters and their applications.

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