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A BRIEF SURVEY – RELATING GRAPH DOMINATION, PLANARITY ON GRAPH BOUNDS

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Abstract

A dominating set of G is a set of vertices of G such that every vertex in V – D is adjacent to a vertex in D. The domination number of G, denoted by γ (G), is the minimum cardinality of a dominating set. Planar graph is a graph that can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. One of the fascinating areas of graph theory is attempting to relate various graph parameters together. In this article, we present a brief survey on graph bounds relating graph domination and planar graphs.

Key Words: Graph Bounds, Domination, Domination Number, Outer Planar Graphs, Planar graphs.

Introduction

The study of domination number in graph theory was introduced by Claude Berge in 1958, on his book "Theory of Graphs and its Applications" [1]. But domination number was studied in the name of coefficient of external stability. At the first time the name dominating set and domination number was used by Oyestein Ore in [2]. At the beginning stage the domination number was denoted by d (G).

Later, in 1977, the notation γ (G) is used to denote the domination number in [3], by E. J. Cockayne and S. T. Hedetniemi. The literature on domination has been surveyed in detail in two famous books by T.W. Haynes, S.T. Hedetniemi, and P.J. Slater [4][5].

Domination theory is now well established and is spreading its contributions to different domains in graph theory and graph applications. Since the evolution of domination theory, types of domination are also established and studied by many researchers.

In this survey we concentrate on planar graphs and domination number. But we restrict our survey on graph bounds relating planar graphs and graph domination.

Graph Theory Terminology and Concepts

All graphs in this paper are undirected and simple. Let G = (V, E) be a graph with the vertex set V of order | V (G) | = n and edge set E of size | E | = m, and let v be a vertex in V. P_n, C_n, K_n, denotes the path, cycle and complete graph with n vertices respectively. The wheel graph is denoted by W_n, and it is defined as $K_1 + C_{n-1}$. The minimum and maximum degree on the vertices of G is denoted by δ (G) and Δ (G) respectively. If δ (G) = Δ (G) = r, then all points have the same degree and G is called regular of degree r. The regular graphs are those of degree 3, such graphs are called cubic.

The degree of a vertex v in G is denoted by d (v). For a vertex subset S $\subseteq$ V(G), define N(S) = { x $\in$ V(G) \ S | there is a y $\in$ S such that xy $\in$ E (G) }. When S = {v}, we write N (v) = N (S) for short. The distance between two vertices u and v in G is denoted by d (u, v). The girth g (G) of G is the length of a shortest cycle in G. If G has no cycles we define the girth of G to be infinite.

A cut vertex of a graph is one whose removal increases the number of components, that is if v is a cut vertex of a connected graph G , then G – { v } is disconnected. The Cartesian product G□H of graphs G and H is a graph such that

- the vertex set of G □ H is the Cartesian product V(G) × V(H); and
- any two vertices (u, u') and (v, v') are adjacent in G □ H if and only if either
    - u = v and u' is adjacent with v' in H, or
    - u' = v' and u is adjacent with v in G.

Let G – v (respectively, G – e) denote the graph formed by removing vertex v (respectively, removing an edge e) from G. Let G + e denote the graph formed by adding an edge e, where e $\in$ E ($\bar{G}$).

A planar graph is a graph that can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. A 1- planar graph is a graph which can be drawn on the plane so that every edge crosses at most one other edge.

A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face. We usually choose this face to be the exterior face. For an inner face f of G, f is said to be an internal triangle if it is not adjacent to the exterior face. An outerplanar graph G is maximal outerplanar if no line can be added without losing outerplanarity. The dual graph of a plane graph G is a graph that has a vertex corresponding to each face of G, and an edge joining two neighboring faces for each edge in G. For details of on graph theory we refer to [6].

A dominating set, denoted by DS, of G is a set of vertices of G such that every vertex in V – D is adjacent to a vertex in D. Such a dominating set D is called a connected dominating set if the subgraph induced by D, $\langle D \rangle$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS. The cardinality of any minimum dominating set (MDS) for G is called the domination number of G and it is denoted by $\gamma(G)$. γ - set denotes a dominating set for G with minimum cardinality. The set D is a distance k-vertex cover of G if for each edge e of G, there is a path of length at most k that contains e and a vertex from D. A connected domatic partition of G is a partition of the vertex set V, into connected dominating sets. The maximum number of subsets in such a partition is called the connected domatic number of G and is denoted by $d_c(G)$. For details of on domination we refer to [4].

Domination and Planarity

Planar graph had always been of attraction to graph theorist due to its complexity of classification and difficulty in characterization. A dominating set determines the adjacency between vertices, while planarity relates on non – crossing of edges. Relating these two parameters is explored extensively by various researchers. In such cases, bondage number plays an important role as it relates removal of edges which results in change of the domination number. Removal of edges in any graph provides better scope for graph planarity. In this view, several results relating bondage number and planarity are established. In this survey, we focus to relate planarity with bondage number, branch width, degree. Also results relating to outer planarity are also discussed. In this brief survey we stick onto the following ordering.

- Bondage Number
- Branch – width
- Degree
- Outer planarity
- Planar graph characterization

In the process of survey since large number of results are established by various researchers on graph bounds, we restrict our survey to graph bounds only. Several results are available in this regard, and many have been omitted in this brief survey. We apologize to the authors for the omission.

Bondage Number

Graph theorists have used bondage number, to investigate planar graphs. This is due to the fact that removal of some edges may contribute to planarity of a graph. In this section, we list a few of the results relating bondage number and planarity.

Fink et.al [7] introduced the bondage number of a graph in 1990. The bondage number b(G) of a nonempty undirected graph G is the minimum number of edges whose removal from G results in a graph with larger domination number. The precise definition of the bondage number is defined as follows.

$$b (G) = \min\{ | B | : B \subseteq E(G), \gamma (G - B) > \gamma (G) \}.$$

Since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma (G)$, the bondage number of a nonempty graph is well defined. We call such an edge set B that $\gamma (G - B) > \gamma (G)$, the bondage set and the minimum one the minimum bondage set. In fact, if B is a minimum bondage set, then $\gamma (G - B) = \gamma (G) + 1$, because the removal of one single edge cannot increase the domination number by more than one. If b (G) does not exist, for example for null graphs, we define $b (G) = \infty$.

Teschner [8] showed that $b (G) \leq \frac{3}{2} \Delta (G)$ holds for any graph G satisfying $\gamma (G) \leq 3$. Furthermore, the following new conjecture was posed.

Conjecture [8] If G is a planar graph, then $b (G) \leq \Delta(G) + 1$.

In 2000, Liying Kanga, Jinjiang Yuan proved that, if G is a connected planar graph, then

$b(G) \leq \min \{8, \Delta(G) + 2\}$[9].

Theorem [9] If G is a connected planar graph, then $b (G) \leq 8$.

Theorem [9] If G is a connected planar graph and there is no degree five vertex then $b (G) \leq 7$.

Theorem [9] If G is a connected planar graph, then $b (G) \leq \Delta (G) + 2$.

In 1998, J.E. Dunbar, T.W. Haynes, U. Teschner, and L. Volkmann posed the conjecture that $b (G) \leq \Delta (G) +1$ for every nontrivial connected planar graph G. Two years later, L. Kang and J. Yuan proved $b(G) \leq 8$ for every connected planar graph G, and therefore, they confirmed the conjecture for $\Delta (G) \geq 7$.

In 2003 Miranca Fischermann, Dieter Rautenbach, Lutz Volkmann proved that this conjecture is valid for all connected planar graphs of girth $g(G) \geq 4$ [10].

Theorem [10] Let G be a nontrivial connected planar graph, if

- $g(G) \geq 4$, then $b(G) \leq 6$.
- $g(G) \geq 5$, then $b(G) \leq 5$.
- $g(G) \geq 6$, then $b(G) \leq 4$
- $g(G) \geq 8$, then $b(G) \leq 3$

Corollary [10] Let G be a nontrivial connected planar graph with $g(G) \geq 5$. If G is not 3-regular, then $b(G) \leq \Delta(G) + 1$.

In 2010, Qiaoling Ma, Sumei Zhang, Jihui Wang**,** proved that $b(G) \leq 12$ for a 1-planar graph G [11].

Theorem [11] If G is a 1-planar graph, then $b(G) \leq 12$.

Theorem [11] If G is a 1-planar graph and there is no degree seven vertex, then $b(G) \leq 11$.

In 2011, Nader Jafari Rad, Lutz Volkmann presented different bounds on the Roman bondage number of planar graphs [12].

Roman bondage number $b_R(G)$ of a graph G with maximum degree at least two is the cardinality of a smallest set of edges $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$.

Theorem [12] If G is a connected planar graph of order $n \geq 3$, then $b_R(G) \leq 2\Delta(G)$.

- $b_R(G) \leq 2\Delta(G)$.
- $b_R(G) \leq \Delta(G) + 6$.

Theorem [12] Let G be a connected planar graph of order $n \geq 3$. If $g(G) \geq 4$, then

- $g(G) \geq 4$, then $b_R(G) \leq \Delta(G) + 4$.
- $g(G) \geq 5$, then $b_R(G) \leq \Delta(G) + 3$.
- $g(G) \geq 6$, then $b_R(G) \leq \Delta(G) + 2$.
- $g(G) \geq 8$, then $b_R(G) \leq \Delta(G) + 1$.

Theorem [12] If G is a connected planar graph of order at least three without vertices of degree five, then $b_R(G) \leq \Delta(G) + 5$.

In 2013, Saieed Akbari, Mahdad Khatirinejad, Sahar Qajar showed that the Roman bondage number of every planar graph does not exceed 15 and constructed infinitely many planar graphs with Roman bondage number equal to 7 [13].

Theorem [13] For every planar graph P, $b_R(P) \leq 15$.

They claim that the upper bound is not possible. In [13] they define a graph operation

Operation: For a graph G of order n, let $\widehat{G}$ be the graph of order 5n obtained from G by attaching the central vertex of a copy of P_5, to each vertex of G.

Using this they have proved that

Corollary [13] There exist infinitely many planar graphs P with $b_R(P) = 7$.

Conjecture [13] The Roman bondage number of every planar graph is at most 7.

In 2012, Jia Huang, Jun-Ming Xu considered some conjectures on the bondage number of a planar graph, and showed limitations of known methods and presented some new approaches to the conjectures by investigating the effects of edge deletion and contraction on the bondage number. To prove this, they have provided two basic upper bounds of b (G) [14].

Lemma [14] $b(G) \le d_G(x) + d_G(y) - 1$ for any two distinct vertices x and y with $d_G(x, y) \le 2$ in G.

Lemma [14] $b(G) \le d_G(x) + d_G(y) - 1 - |N_G(x) \cap N_G(y)|$ for any two adjacent vertices x and y in G.

In 2012 Jia Huang, Jun-Ming X devised a new approach to discuss the conjectures [25, 9, 10]. A basic way is to find two vertices x and y in G satisfying the conditions in Lemma [14], such that the value bounded b(G), $d_G(x) + d_G(y)$ or $d_G(x) + d_G(y) - |N_G(x) \cap N_G(y)|$ is as small as possible. Precisely, let

$$B(G) = \min_{x,y\,\in V(G)} \begin{cases} \{ d_G(x) + d_G(y) - 1 : & 1 \le d_G(x, y) \le 2\} \cup \\ \{d_G(x) + d_G(y) - 1 - |NG(x) \cap NG(y)| : & d_G(x, y) = 1\} \end{cases}$$

Then by Lemma [14], we have $b(G) \le B(G)$.

Theorem [14] $B(G) \le \min\{8, \Delta(G) + 2\}$ for any connected planar graph G.

Theorem [14] For any connected planar graph G,

$$B(G) \le \begin{cases} 6, & \text{if } g(G) \ge 4; \\ 5, & \text{if } g(G) \ge 5; \\ 4, & \text{if } g(G) \ge 6; \\ 3, & \text{if } g(G) \ge 8; \end{cases}$$

They [14] have disproved the following conjectures, using edge subdivision. They [14] have provided counter examples for the same.

Conjecture [14] $B(G) \le \Delta(G) + 1$ for any planar graph G.

Conjecture [14] $B(G) \le 7$ for any connected planar graph G.

Conjecture [14] $B(G) \le 5$ for any connected planar graph G with $g(G) \ge 4$.

Conjecture [14] $B(G) \le 4$ for any connected planar graph G with $g(G) \ge 5$.

They [14] in fact have provided better bounds in the following theorems.

Theorem [14] There are planar graphs G with B (G) = 5,6, 8.

Given a graph G, the contraction of G by the edge e = (x, y), denoted by G/xy, is the graph obtained from G − e by replacing x and y with a new vertex v_{xy} (contracted vertex).

In [14], they have further proved that

Lemma [14] Let e be an edge of G. Then $b (G – e) \geq b (G) −1$. In addition, $b (G – e) \leq b (G)$ if $\gamma (G − e) = \gamma(G)$.

Lemma [14] $\gamma (G) − 1 \leq \gamma (G /xy) \leq \gamma (G)$ for any edge xy of G.

Theorem [14] $b (G_i – e) = b(G_i) − 1$ for any edge e in G_i, i = 1, 2, 3, 4.

Theorem [14] $b (G / xy) \geq b (G)$ if $\gamma (G / xy) = \gamma (G)$ and $N_G (x) \cap N_G (y) = \phi$ for some edge(x, y) of G.

Theorem [14] $\gamma (G_1 /xy) = \gamma (G_1) − 1$ for every edge xy.

In 2001 Bert L. Hartnell, Halifax, and Douglas F. Rall, Greenville established a sharp lower bound on the number of edges in a connected graph with a given order and given connected domatic number and also showed that a planar graph has connected domatic number at most 4 and characterized planar graphs having connected domatic number 3 [15].

Theorem [15] Let $2 \leq r \leq s$, $G_{r,s} = P_r \square P_s$, (P_r be the path $v_1, v_2, . . . , v_r$ and let P_s be the path $w_1, w_2, . . . , w_s$) the Cartesian product of two paths. If r = 2, then $d_c (G_{r,s}) = 2$. For $r \geq 3$, $dc (G_{r,s}) = 1$.

Theorem [15] Let G be a planar graph. The connected domatic number of G is at most 4, and K_4 is the only planar graph achieving this bound.

Theorem [15] Let G be a planar graph such that $d_c (G) = 3$ and let $D_1 \cup D_2 \cup D_3$ be any connected domatic partition of G. Each of the induced subgraphs $\langle D_1 \rangle$, $\langle D_2 \rangle$ and $\langle D_3 \rangle$ is a path.

Branch-Width

Branch-width was introduced by Robertson and Seymour in their graph minors series of papers several years after tree-width. These parameters are rather close, but surprisingly many theorems of the graph minors series are easier to prove when one uses branch-width instead of tree-width.

In 2006 Fedor V. Fomin, Dimitrios M. Thilikos introduced a new approach to design parameterized algorithms on planar graphs and also showed that the k- dominating set problem on planar graphs can be solved in time $O (2^{15.13\sqrt{k}} + n^3)$ in the following results [16].

Σ - plane graph

Let Σ be a sphere. By Σ - plane graph G we mean a planar graph G with the vertex set V (G) and the edge set E (G) drawn in Σ. If $\Delta \subseteq \Sigma$, then Δ denotes the closure of Δ, and the boundary of Δ is $\hat{\Delta} = \bar{\Delta} \cap \overline{\Sigma - \Delta}$.

A D - dominated graph G is uniquely dominated if there is no path of length < 3 connecting two vertices of D.

Lemma [16] Let G be a planar graph with a dominating set of size ≤ k. Then bw (G) ≤ $12\sqrt{k} + 9$.

Theorem [16] For every 2-connected D-dominated Σ - plane graph G without multiple edges, there exists a Σ - plane graph H such that the following hold:

(a) G is a minor of H.

(b) H is uniquely D - dominated.

(c) All multiple edges of H are exceptional.

(d) For any face r of H, boundary $\hat{r}$ is either a triangle or a square.

(e) If the distance between vertices x, y ∈ D in H is three, then there exist atleast two distinct (x, y) - Paths in H of length three.

(f) If a (closed) face r of H contains a vertex of D, then $\hat{r}$ is a triangle.

(g) Every square face of H contains two edges e_i, i = 1, 2, without common vertices such that for each i = 1, 2, there exists a vertex $x_i \in D$ adjacent to both endpoints of e_i.

(h) If x, y ∈ D, then every two distinct (x, y) - paths of H of length three are internally disjoint.

A connected D-dominated Σ-plane graph G, satisfying properties (b) – (h) of above Lemma, are said to be nicely D - dominated Σ-plane graphs.

Lemma [16] Let G be a nicely D-dominated Σ-plane graph G and let T ∈ τ (G) be a nonempty D-triangle bounding the closed discs Δ_1, Δ_2. Let also G_i, i = 1, 2, be the subgraph of G containing all vertices and edges included in Δ_i. Then G_i, i = 1, 2, is a nicely D_i - dominated graph for some $D_i \subseteq D$ and G_i has fewer vertices than G.

Lemma [16] Let G be a nicely D-dominated Σ- plane graph G and let C = (x, a, b, y, c, d, x) be a non - empty D-hexagon with poles x, y bounding the closed discs Δ_1, Δ_2. Let also G_i, i = 1, 2, be the graph containing all the edges and vertices included in Δ_i and extended by adding the edges {b, c} and {a, d} (edges {b, c} and {a, d} are placed outside D_i to ensure planarity of G_i). Then G_i, i = 1, 2, is a nicely D_i dominated graph for some $D_i \subseteq D$ and G_i, i = 1, 2, has fewer vertices than G.

Prime D-dominated Σ-plane graphs. A nicely D-dominated Σ-plane graph G is a prime D -dominated Σ-plane graph (or just prime) if all its D-triangles and D-hexagons are empty.

T (G) is the set of all the triangles (cycles of length three) containing a vertex of D. The triangles in T (G) is said to be D-triangles. C (G) is the set of all cycles consisting of two distinct paths of length three connecting two vertices of D. The cycles in C (G) is said to be D-hexagons.

Lemma [16] Let G be a prime D-dominated Σ-plane graph. If G contains two vertices x, y ∈ D connected by three paths of length three, then $V(P_1) \cup V(P_2) \cup V(P_3) = V(G)$.

Lemma [16] Let G be a prime D-dominated Σ-plane graph with $|D| \geq 2$. For any D - triangle T = (x, a, b) with x ∈ D, the edges {x, a} and {x, b} are also the edges of some D-hexagon of G with poles x and y ∈ D. Moreover, if $|D| \geq 3$, the edge {a, b} is in Δ ({x, y}).

Lemma [16] Let G be a prime D - dominated Σ-plane graph with $|D| \geq 2$. Then the endpoints of each edge of G are the vertices of some D - hexagon.

Lemma [16] For any nicely D - dominated Σ-plane graph G, $bw(G) \leq \sqrt{3.4.5.|D|}$.

Theorem [16] Let G be a D - dominated Σ-plane graph. Then $bw(G) \leq 3\sqrt{4.5.|D|}$.

Lemma [16] There exist planar graphs with a dominating set of size ≤ k and with bw > 3√k.

Degree

In 1998, J.E. Dunbar, T.W. Haynes, U. Teschner, and L. Volkmann posed the conjectured that $b(G) \leq \Delta(G) + 1$ for every nontrivial connected planar graph G. Two years later, L. Kangand J. Yuan proved $b(G) \leq 8$ for every connected planar graph G, and therefore, they confirmed the conjecture for $\Delta(G) \geq 7$.

In 2003 Miranca Fischermann, Dieter Rautenbach, Lutz Volkmann proved that this conjecture is valid for all connected planar graphs of maximum degree $\Delta(G) \geq 5$ [10].

For a graph G, let $n_i(G) = n_i$ be the number of vertices of degree i and $\tau_i(G) = \tau_i$, be the number of vertices of degree at least i for i=1, 2, …, Δ (G).

Theorem [10]

Let G be a connected planar graph, and let A be the vertices of degree 5 which have distance at least three to the degree 1, 2, and 3 vertices. If all vertices in A, which are not adjacent with degree 4 vertices, are independent and not adjacent to degree 6 vertices, then $b(G) \leq 7$.

Theorem [10] Let G be a connected planar graph. If there are two vertices u and v such that d (u, v) ≤ 2 and d (u) + d (v) ≤ 8, or if $n_5 < 2n_2 + 3n_3 + 2n_4 + 12$ then b (G) ≤ 7.

Proposition [10] If G is a connected planar graph without degree four and degree five vertices, then b (G) ≤ 6.

Proposition [10] Let G be a nontrivial connected planar graph with Δ (G) = 6. If every edge e = (x, y) with d (x) = 5 and d(y) = 6 is contained in at most one triangle, then b (G) ≤ 7 = Δ (G) + 1.

Corollary [10] Let G be a nontrivial connected planar graph with Δ (G) ≥ 6. If Δ (G) ≤7 or if Δ (G) = 6 and every edge e = (x, y) with d (x) = 5 and d (y) = 6 is contained in at most one triangle, then b (G) ≤ Δ (G) + 1.

Corollary [10] Let G be a nontrivial connected planar graph with Δ (G) = 5. If no triangle contains an edge e = (x, y) with d (x) = 5 and 4 ≤ d (y) ≤ 5, then b (G) ≤ 6 = Δ (G) + 1.

When graph theorists speak of planarity, automatically discussion on outer planarity also comes into picture by default. In this section we present some results relating dominating sets and outer planarity of graphs

Outer Planar

In 2000 Liying Kanga, Jinjiang Yuan proved the following result [9].

Theorem [9] If a connected graph G has no minor isomorphic to K_4, then b (G) ≤ 3 and this result is the best possible.

Theorem [9] If G is an outer-planar graph, then b (G) ≤ 3.

In 1996, Matheson and Tarjan conjectured that any n-vertex plane triangulation with n sufficiently large has a dominating set of size atmost $^{n}/_{4}$.

In 2010 Erika L.C. King, Michael J. Pelsmajer proved the conjecture of Matheson and Tarjan for the graphs of maximum degree 6 [17].

Theorem [17] There exists n_0 such that for any $n \geq n_0$, an n-vertex triangulation with maximum degree 6 has a dominating set of size atmost $^{n}/_{4}$.

In 1996, Matheson and Tarjan proved that any triangulated disc G with n vertices satisfies $\gamma (G) \leq \lfloor \frac{n}{3} \rfloor$. Later Honjo, Kawarabayashi and Nakamoto extended this result to triangulations of other surfaces. Matheson and Tarjan conjectured that $\gamma (G) \leq \frac{n}{4}$ for every n-vertex triangulation G with sufficiently large n. In 2010, King and Pelsmajer proved this conjecture for graphs of maximum degree 6.

In 2013, Shin-ichi Tokunaga proved that G is an n-vertex maximal outerplanar graph with n ≥ 3 having k vertices of degree 2, then G has a dominating set of size at most $\frac{n+k}{4}$, by a simple coloring method [18].

Theorem [18] Suppose G is an n-vertex maximal outerplanar graph with n ≥ 3, having k vertices of degree 2, then γ (G) ≤ $\lfloor \frac{n+k}{4} \rfloor$. To strengthen Matheson and Tarjan's theorem, they proposed the following conjectures for 2-connected planar graphs.

Conjecture [18] Suppose G is a 2-connected planar graph with n vertices such that each of its vertices of degree 2 belongs to a triangle, then γ (G) ≤ $\lfloor \frac{n}{3} \rfloor$.

Conjecture [18] Suppose n ≥ 5 and G is an n-vertex connected graph with minimum degree 2; then γ (G) ≤ $\lfloor \frac{2}{5} n \rfloor$.

Conjecture [18] Suppose G is a 3-connected triangulated disc with n vertices, then γ (G) ≤ $\lfloor \frac{n+2}{4} \rfloor$.

In 2013, C.N. Campos, Y. Wakabayashi proved that if G is an n-vertex maximal outerplanar graph, then γ (G) ≤ (n + t) / 4, where t is the number of vertices of degree 2 in G and showed that this bound is tight for all t ≥ 2[19].

Theorem [19] Let G be a maximal outerplanar graph with k internal triangles and n ≥ 3 vertices. Then, γ(G) ≤ $\frac{n+k+2}{4}$.

Corollary [19] Let G be a maximal outerplanar graph with n ≥ 4 vertices and t vertices of degree 2. Then, γ(G) ≤ $\frac{n+t}{4}$

In 2013 Campos and Wakabayashi proved that γ (G) ≤ $\lfloor \frac{n+k}{4} \rfloor$ for any maximal outerplanar graph G of order n ≥ 3 with k vertices of degree 2. In 2013 Tokunaga provided a short proof for the above theorem. Based on some structural properties of $K_{2,3}$ - minor free graphs and K_4 - minor free graphs and applying the idea of Tokunaga [18], Tingting Zhu, Baoyindureng Wu in 2015 extended the theorem of Campos and Wakabayashi [19] to all maximal K_4 - minor free graphs and all maximal $K_{2,3}$-minor free graphs and disproved the two conjectures of Tokunaga [18] on planar graphs [20].

Theorem [20] Assume that G is a maximal $K_{2,3}$-minor free graph or a maximal K_4 - minor free graph of order n ≥ 3. If k_i is the number of vertices of degree i in G for i = 1, 2, then γ (G) ≤ $\lfloor \frac{n+2k_1+k_2}{4} \rfloor$.

Corollary [20] If G is a maximal K_4-minor free graph of order n ≥ 3 having k vertices of degree 2, then γ (G) ≤ $\lfloor \frac{n+k}{4} \rfloor$.

In 2015, José D. Alvarado, Simone Dantas, Dieter Rautenbach showed that $\gamma_k(n) = g_k(n) = \lfloor \frac{n}{2k+1} \rfloor$ for n ≥ 2k + 1, and $\beta_k(n) = \lfloor \frac{n}{2k} \rfloor$ for n ≥ 2k ≥ 4 [21].

Theorem [21] If k and n are positive integers with $n \geq 2k + 1$, then $\gamma_k(n) = g_k(n) = \lfloor \frac{n}{2k+1} \rfloor$. To strengthen the above Theorems they [21] proved the following result.

Lemma [21] Let k and n be positive integers, $\gamma_k(n + 1) \geq \gamma_k(n)$.

Planar graph characterization

At this stage of the survey, we recollect Kuratowski′s theorem perhaps the best ever result in graph planarity. A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

We also recollect that a graph G is outer planar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - e$. With these two theorems as a basis in the following section, we provide a brief survey on planar graph characterization.

In 2015, M. Yamuna, A. Elakkiya characterized planarity and outer - planarity of the complement $\overline{G}$ of a Non – domination subdivision stable graph (NDSS) using domination number and also provided MATLAB code for identifying NDSS graphs [22].

Non – domination subdivision stable graph

A graph G is said to be non - domination subdivision stable if $\gamma(G_{sd}uv) = \gamma(G) + 1$ for all u, v ∈ V (G), u adjacent to v.

Theorem [22] If G is a NDSS graph such that

$$\gamma(G) = \begin{cases} 4, & \text{then } \overline{G} \text{ is non} - \text{planar.} \\ 3, & \text{then } \overline{G} \text{ is non} - \text{planar.} \\ 2, & \text{then } \overline{G} \text{ is non} - \text{Outer planar.} \end{cases}$$

They [22] also provided matrix representation and MATLAB code for identifying NDSS graphs.

In 2015 M. Yamuna, K. Karthika, determined the domination number of G^*, $\overline{G^*}$, chromatic polynomial of G^*, spanning tree of G^*, number of spanning trees of G^* from G. MATLAB code for determining domination number of G^*, $\overline{G^*}$ is also provided [23].

Theorem [23]

Let G be a graph and G^* be its dual. $\gamma(G^*) = k$ if and only if

i. there is a set D of k- regions in G such that every region in S−D is adjacent to at least one region in D.

ii. there is no set $X \subseteq S$, that satisfies condition 1 such that $| X | < | D |$.

Theorem [23] Let G be a planar graph, $\gamma(G^*) = k$. G^* is γ- stable if and only if

i. there is a set of k- adjacent regions $D \subseteq S$ such that every region in $S - D$ is adjacent to at least one region in D.

ii. every set $D \subseteq S$ such that

- $|D| = k$,
- every region in $S - D$ is adjacent to at least one region in D, is k- adjacent.

Theorem [23] If G and G^* are - stable graphs, then

i. $\gamma(G) + \gamma(G^*) \leq \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{5} \rfloor$;

ii. $\gamma(G) . \gamma(G^*) \leq \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{5} \rfloor$.

Theorem [23] Let G be a planar graph such that its dual G^* is γ - stable. For any γ- set D^* in G^*

i. $(G - a)^* = |D^*| - 1$.

ii. $(G - a)^*$ is γ- stable.

for all $a^* \in \langle D^* \rangle$, $a \in E(G)$.

In 2014, M. Yamuna, K. Karthika have obtained a characterization of planar graphs when G and $\overline{G}$ are γ - stable graphs [24]

γ - stable graphs

A graph G is said to be γ - stable if $\gamma(G_{xy}) = \gamma(G)$, for all $x, y \in V(G)$, x is not adjacent to y, where G_{xy} denotes the graph obtained by merging the vertices x , y.

Theorem [24] If G and $\overline{G}$ are γ - stable graphs such that $\gamma(G) \leq 4$ and $\gamma(\overline{G}) = 4$, then G is non - planar.

They [24] conclude that if G, $\overline{G}$ are γ - stable graphs, then

1. G is non - planar, if $2 < \gamma(\overline{G}) \leq 4$.

2. G need not be non - planar, if $\gamma(\overline{G}) = 2$.

Conclusion

A survey on several papers enlightened the fact that it is impossible to provide a short survey relating all parameters of a graph to planarity and dominating set. Surprisingly the search of results resulted in discovering that, graph theorist have established plenty of results on graph bounds. So in this survey, we have related graph domination and planarity restricting to graph bounds.

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