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GENERATING NEW STRUCTURES ON TOTAL DOMINATION DOT STABLE GRAPHS

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Abstract

A graph G is said to be total domination dot stable (TDDS) if $\gamma_t(G_{\bullet} uv) = \gamma_t(G)$ for all u, v $\in$ V (G). In this paper we have proved a necessary and sufficient condition for a graph to be TDDS and obtained some methods of generating TDDS graphs from existing TDDS graphs.

Key words: domination, total domination, total dot – critical, total domination dot stable.

1. Introduction

Let $G = (V, E)$ be a graph. A set $D \subset V$ is a dominating set of G if every vertex in $V – D$ is adjacent to some vertex in D. If D has the smallest possible cardinality of any dominating set of G is called the minimum dominating set abbreviated *MDS*. The cardinality of any *MDS* for G is called the domination number of G and it is denoted by $\gamma (G)$. A total dominating set is a set S of vertices in a graph G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of the total dominating set of G is the total domination number of G. It is denoted by $\gamma_t(G)$. The boundary of D, denoted by $B (D)$ is $N (D) – D$. Let $v \in V (G)$. The open neighborhood of v denoted by $N (v)$, is defined by the set $N (v) = \{ u / uv \in E (G) \}$ and the closed neighborhood of v denoted by $N [v]$, is defined by $N [v] = N (v) \cup \{ v \}$. The subgraph of G induced by the vertices in D is denoted by $< D >$. A graph G is said to be pairwise independent if $\forall$ $u, v \in D$, $Pn [u, D] \neq \phi$, $Pn [v, D] \neq \phi$ and, $B (\{u, v\}) \cap D = \phi$, where D is a γ_t – *set* for G. A graph G is said to be total excellent if given any vertex v then there is a γ_t – *set* of G containing v. A vertex v is said to be good if there is a γ_t – *set* of G containing v. If there is no γ_t – *set* of G containing v, then v is said to be a bad vertex. In [2] Henning and Rad investigated criticality of the total domination number when the graph is modified by identifying adjacent vertices and also they investigated when identifying changed the domination number. A graph G is total dot - critical if $\gamma_t (G_{\bullet}uv) < \gamma_t (G)$ for any two adjacent vertices v and u. The graph formed from G by identifying u and v is denoted as $G\bullet uv$. $G\bullet uv$ may be viewed as the graph obtained from G by deleting u and v adding a new vertex, denoted by (uv), that is adjacent to all of the vertices of $G-u-v$ that were originally adjacent to either of u or v. In [3] results based on *TDDS* graphs that are total excellent, just total excellent, total very excellent have been obtained. In [4] Stephanie A. Rickett, T.W. Haynes introduced the concept of total domination dot stable graphs. A graph is total domination dot stable if identifying any pair of adjacent vertices the total domination number unchanged. ie., $\gamma_t(G\bullet uv) = \gamma_t(G), \forall u, v \in V(G), u \perp v$. The following results have been proved in [4].

1. Let G be a connected graph of order $n \geq 3$. For any two vertices a and b, $\gamma_t(G) - 2 \leq \gamma_t(G\bullet ab) \leq \gamma_t(G)$.
2. A graph G is $\gamma_t - dot - stable$ iff $\forall\ \gamma_t(G) - sets\ S$ the induced subgraph $G < S >$ is a set of independent edges.
3. For any graph G with no isolates $\gamma_t(G) \leq \frac{2n}{3}$.

2. Total Domination Dot Stable Graph

Definition: A graph G is said to be total domination dot stable *(TDDS)* if identifying any pair of adjacent vertices the total domination number unchanged. ie., $\gamma_t(G\bullet uv) = \gamma_t(G), \quad \forall u, v \in V(G)$ such that $u \perp v$.

Example for *TDDS*: Path P_n is *TDDS* iff $\gamma_t(P_n) = \gamma_t(P_{n-1})$. C_n is *TDDS* iff $\gamma_t(C_n) = \gamma_t(C_{n-1})$.Complete graph K_n, Complete bipartite graph $K_{m,n}$, and Star graph S_n, is also *TDDS*. The graph G given in **Fig. 2.1** is *TDDS*.

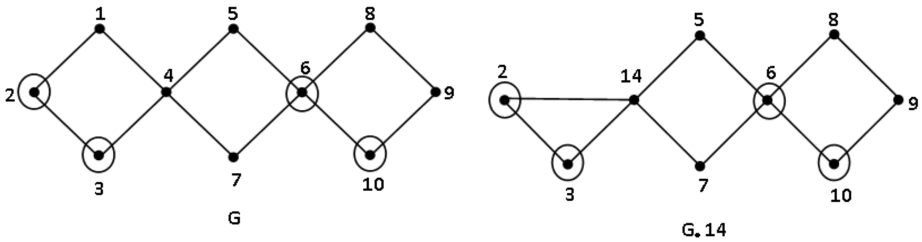

Fig. 2.1

Consider the above graphs, Here $\gamma_t(G) = 4 = \gamma_t(G\bullet 14)$. This is true $\forall\ u, v \in V(G)$. ie., $\gamma_t(G\bullet uv) = \gamma_t(G)$. G is *TDDS*.

Theorem 2.1

G is TDDS iff $\forall u, v \in D$, $B(\{u, v\}) \cap D = \phi$, where D is a $\gamma_t - set$ for G.

Proof

Let G be *TDDS*. If $B(\{u, v\}) \cap D \neq \phi$ then for $u, v \in D$, $\exists$ one $w \in B(\{u, v\})$ such that $w \perp u$ or $w \perp v$. If $w \perp u$ then $u, v, w \in D$ such that $u \perp v$, $u \perp w$. $\gamma_t(G\bullet uw) = \gamma_t(G) - \{u\} - \{w\} \cup \{uw\}$. [All vertices dominated by u and w is now dominated by uw].

ie., $\gamma_t (G_\bullet uw) < \gamma_t (G)$, which is a contradiction as G is *TDDS.*

Conversely, We assume that $B (\{ u, v \}) \cap D = \phi$. If G is not *TDDS* then by result [R_1] either u or v is $\gamma_t - critical$ or for $u, v \in D$ such that $B (\{ u, v \}) \cap D \neq \phi$. But $B (\{ u, v \}) \cap D \neq \phi$ is not possible by our assumption. u or v may be critical. Let us assume that u is critical.Let $H = G - \{ u \}$, $\gamma_t (H) = \gamma_t (G) - 1$. Let D' be a $\gamma_t - set$ for H.

Case 1

If $N (u) \not\subseteq D'$. In D', $N (u)$ is total dominated. ie., $\exists\ x, y \in D'$ such that $x \perp y, y \perp v$, where $v \in N (u)$. Now $D'' = D' \cup \{ v \}$ is a $\gamma_t - set$ for G such that $B (\{ x, y \}) \cap D = \{ v \}$, which is contradiction to our assumption.

Case 2

If $\exists$ atleast one vertex in $N (u) \in D'$. Let $v \in N (u)$ such that $v \in D'$. Let $w \in D'$ such that $w \perp v$. Then $D'' = D' \cup \{ u \}$ is a $\gamma_t - set$ for G such that $w \perp v \perp u$, ie., $B (\{ u, v \}) \cap D = \{w \}$, which is contradiction to our assumption. Thus either u or v is not critical. Hence G is *TDDS*. □.

Corollary

1. *If G is TDDS, then* $\gamma_t (G) \neq 3$.

Proof

Since G is total dominated if $\gamma_t (G) = 3$ say $\gamma_t (G) = u, v, w$, then u adjacent to v adjacent to w, ie., $B (\{u, v\}) \cap D = \{w\}$. Hence G is not *TDDS.* □

2. *If G is a pairwise independent, then G is TDDS.*

Proof

G is pairwise independent implies $B (\{u, v\}) \cap D = \phi\ \forall u, v \in D$. Hence by theorem 2.1 G is *TDDS*. □

3. *A TDDS graph cannot have a down vertex.*

Proof

Let G be *TDDS* graph. Let D be a $\gamma_t - set$ of G. Let $u, v \in V (G)$. If u is critical then

$\gamma_t (G_\bullet uv) < \gamma_t (G)$. By converse part of theorem 2.1, $B (\{u, v\}) \cap D = \{x\}$ for any $x \in D$ such that x adjacent to v which is a contradiction as G is *TDDS.*

3. Constructions

In this section we show that every graph is an induced subgraph of a *TDDS* graph and discuss few methods of generating new *TDDS* graphs from existing *TDDS* graphs.

Theorem 3.1

Every graph is an induced subgraph of a TDDS graph.

Proof

Let G be a any graph with n vertices say x_i, where $i = 1, 2, ..., n$. Consider n copies of P_3 and label the vertices in P_3 as $u_i\, v_i\, w_i\, z_i$ where $i = 1, 2, ..., n$. Obtain a new graph H by merging $x_i\, u_i$ as y_i, $i = 1, 2, ..., n$. We have $\{ v_1, v_2, ..., v_n \} \cup \{ w_1, w_2, ..., w_n \}$ is the only $\gamma_t - set$ for H. $| \gamma_t (H) | = 2n$.

Consider $H_\bullet w_i\, z_i$, $\gamma_t (H) = \gamma_t (H_\bullet\, w_i\, z_i)\ \forall i =1, 2, ..., n$. Since $\{ y_1, y_2, ..., y_n \} \cup \{ v_1, v_2, ..., v_n \}$ or $\{ v_1, v_2, ..., v_n \} \cup \{ w_1 z_1, w_2 z_2, ..., w_n z_n \}$ are $\gamma_t - set$ for $H_\bullet\, w_i\, z_i$. Consider $H_\bullet\, v_i\, w_i$, $\gamma_t (H) = \gamma_t (H_\bullet\, v_i\, w_i)\ \forall i =1, 2, ..., n$. Since $\{ y_1, y_2, ..., y_n \} \cup \{ v_1 w_1, v_2 w_2, ..., v_n w_n \}$ or $\{ v_1 w_1, v_2 w_2, ..., v_n w_n \} \cup \{ z_1, z_2, ..., z_n \}$ are $\gamma_t - set$ for $H_\bullet\, v_i\, w_i$.

Consider $H_\bullet\, y_i\, v_i$, $\gamma_t (H) = \gamma_t (H_\bullet\, y_i\, v_i)\ \forall\ i =1, 2, ..., n$. Since $\{ y_1 v_1, y_2 v_2, ..., y_n v_n\} \cup \{ w_1, w_2, ..., w_n \}$ or $\{ w_1, w_2, ..., w_n \} \cup \{ z_1, z_2, ..., z_n \}$ are $\gamma_t - set$ for $H_\bullet\, y_i\, v_i$.

Consider $H_\bullet y_i\, y_j$, $\gamma_t (G) = \gamma_t (H_\bullet y_i\, y_j)$, $i \neq j$, $i, j = 1, 2, ..., n$. $\gamma_t (H_\bullet uv) = \gamma_t (G)\ \forall\ u, v \in V (H)$. Hence H is *TDDS.*

Theorem 3.2

Let G be a TDDS and total excellent. The graph H is obtained by attaching P_1 to a vertex in G is TDDS.

Proof

Let G be *TDDS* and $D = \{ u_1, u_2, ..., u_k \}$ be $\gamma_t - set$ for G. Obtain a new graph H by attaching a pendant vertex v to a vertex in D say u_i. Now $\gamma_t (H_\bullet\, u_i\, v) = \gamma_t (H)$ as u_i dominates v. $\gamma_t (H_\bullet\, u_i\, u_j) = \gamma_t (H)$. By the observation (It has been proved in [5]) we know that $\exists$ a $\gamma_t - set$ including u_i. Hence H is *TDDS*. □

Remark

1. *If u is a bad vertex and we attach a pendant vertex v at u then the resulting graph H is not TDDS.*

Proof: Since u is bad $\exists x, y \in D, x \perp y$ and $u \perp x$. $D' = \gamma_t (G) \cup \{ u \}$ is a $\gamma_t - set$ for H. ie., $\exists$ a $\gamma_t - set$ such that $x \perp y \perp u$ which implies that H is not *TDDS.*

Example

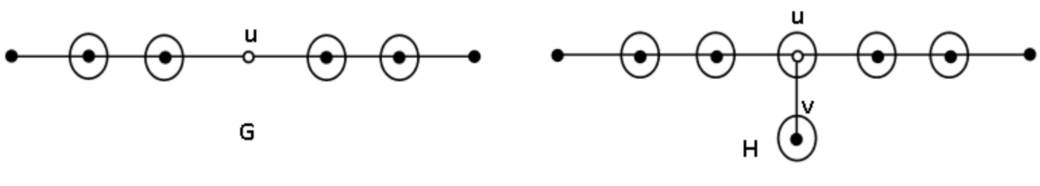

Fig. 2.2

In **Fig. 2.2,** $\gamma_t(G) = 4$, $\gamma_t(H) = 6$. Here $H \bullet uv = G$. ie., $\gamma_t(H \bullet uv) = 4$. $\gamma_t(H) \neq \gamma_t(H \bullet uv)$.

2. *Let G be a TDDS, the graph H is obtained by attaching a path P_2 to any vertex in G, then H is not TDDS.*

Theorem 3.3

Let G be TDDS and u be a bad vertex. Let $G \bullet uv$ be TDDS. Then graph H is obtained by attaching a path P_3 to u is TDDS.

Proof

Let G be pairwise independent graph then by corollary of theorem 2.1 G is *TDDS* and D be a $\gamma_t -$ *set* of G. Let $u \in V(G)$ and $u \notin \gamma_t(G)$. Consider a path $P_3 : w_1\, w_2\, w_3\, w_4$. Attach a path P_3 to a vertex u in G then we obtained a new graph H and label the vertex uw_1 as y. We have $\gamma_t(H) = \gamma_t(G) + \{w_2 \cup w_3\}$. ie., $\gamma_t(H) = \gamma_t(G) + 2$.

Consider $(H \bullet yw_2)$, we have $\gamma_t(G) + \{w_3 \cup w_4\}$ is a $\gamma_t -$ *set* for $H \bullet yw_2$. $\gamma_t(H \bullet yw_2) = \gamma_t(H)$. Consider $(H \bullet w_2\, w_3)$, we have $\gamma_t(G) + \{w_2\, w_3 \cup w_4\}$ is a $\gamma_t -$ *set* for $H \bullet w_2\, w_3$. $\gamma_t(H \bullet w_2\, w_3) = \gamma_t(H)$. Consider $(H \bullet w_3\, w_4)$, we have $\gamma_t(G) + \{w_2 \cup w_3\, w_4\}$ is a $\gamma_t -$ *set* for $H \bullet w_3\, w_4$. $\gamma_t(H \bullet w_3\, w_4) = \gamma_t(H)$. By theorem (*Let G be TDDS and let u be a bad vertex. Let $G \bullet uv$ be TDDS. In $G \bullet uv$ $\exists$ no vertex w such that uv, w $\in$ D, uv $\perp$ w, w is selfish.* It has been proved in [5]) for any $u, v \in G$ such that $u \perp v$. In $G \bullet uv$ if u is good then there is no selfish vertex in $N(u)$. Also $\gamma_t(H \bullet uv) = \gamma_t(H)\ \forall\ u, v \in V(G)$. Hence H is *TDDS.* □

Remark

Let G be TDDS. The graph H is obtained by attaching a path P_4 to $u \notin \gamma_t(G)$ in G then H is TDDS.

Theorem 3.4

Let G be TDDS. Then the graph H is obtained by attaching a path P_5 to a vertex in $\gamma_t -$ set of G is TDDS.

Proof

Let G be *TDDS* and D be a $\gamma_t -$ *set* of G. Let $u \in \gamma_t(G)$. Consider a path $P_5 : w_1\, w_2\, w_3\, w_4\, w_5\, w_6$. Attach a path P_5 to a vertex u in G then we obtained a new graph H and label the vertices uw_1 as y. We have $\gamma_t(H) = \gamma_t(G) + \{w_4 \cup w_5\}$. ie., $\gamma_t(H) = \gamma_t(G) + 2$.

Consider $(H \bullet yw_2)$, we have $\gamma_t(G) + \{w_4 \cup w_5\}$ or $\gamma_t(G) + \{w_5 \cup w_6\}$ is a $\gamma_t -$ *set* for $H \bullet yw_2$. $\gamma_t(H \bullet yw_2) = \gamma_t(H)$. Consider $(H \bullet w_2\, w_3)$, we have $\gamma_t(G) + \{w_4 \cup w_5\}$ or $\gamma_t(G) + \{w_5 \cup w_6\}$ is a $\gamma_t -$ *set* for $H \bullet w_2\, w_3$. $\gamma_t(H \bullet w_2\, w_3) = \gamma_t(H)$. Consider $H \bullet w_3\, w_4$, we have $\gamma_t(G) + \{w_3 w_4 \cup w_5\}$ or $\gamma_t(G) + \{w_5 \cup w_6\}$ is a $\gamma_t -$ *set* for $H \bullet w_3\, w_4$. $\gamma_t(H \bullet w_3\, w_4) = \gamma_t(H)$. Consider $(H \bullet w_4\, w_5)$, we have $\gamma_t(G) + \{w_3 \cup w_4 w_5\}$ or $\gamma_t(G) + \{w_4\, w_5 \cup$

w_6 } is a $\gamma_t - set$ for $H_\bullet w_4 w_5$. $\gamma_t(H_\bullet w_4 w_5) = \gamma_t(H)$. Consider ($H_\bullet w_5 w_6$), we have $\gamma_t(G) + \{w_3 \cup w_4\}$ or $\gamma_t(G) + \{w_4 \cup w_5 w_6\}$ is a $\gamma_t - set$ for $H_\bullet w_5 w_6$. $\gamma_t(H_\bullet w_5 w_6) = \gamma_t(H)$ also $\gamma_t(H_\bullet uv) = \gamma_t(H)\ \forall\ u, v \in V(G)$. Hence *H* is *TDDS.*

Theorem 3.5

Let G_1 and G_2 be TDDS graph. Let $u \in V(G_1)$ and $v \in V(G_2)$. Let H be a graph generated by adding an edge between u and v then H is TDDS if, both u and v are bad vertices.

Proof

Let G_1 and G_2 be *TDDS* graph. Let $u \in V(G_1)$ and $v \in V(G_2)$ be a bad vertices. Let D_1 and D_2 be a dominating for G_1 and G_2. *H* is obtained by adding an edge between *u* and *v*. $\gamma_t(H) \leq \gamma_t(G_1) + \gamma_t(G_2)$. If $\gamma_t(H) < \gamma_t(G_1) + \gamma_t(G_2)$. Then $u \in \gamma_t(H)$ or $v \in \gamma_t(H)$ or $u, v \in \gamma_t(H)$.

Case 1 $u \in \gamma_t(H)$

If $u \in \gamma_t(H)$, then since *u* is a bad vertex for G_1, $\gamma_t(G_1)$ cannot be reduced. Also *u* dominates *v*. $\gamma_t(H) < \gamma_t(G_1) + \gamma_t(G_2)$ ie., If *v* need not be dominated then the $\gamma_t - value$ for G_2 is reduced. Let D' be a $\gamma_t - set$ for G_2 such that D' need not dominate vertex *v*. If $N(v) \in D'$ then $D'' = D' \cup \{v\}$ is a $\gamma_t - set$ for G_2 such that $B(\{u, v\}) \cap D'' \neq \phi$, which is a contradiction as G_2 is *TDDS.* Suppose $N(v) \notin D'$ then $D''' = D' \cup \{w\}$ is a $\gamma_t - set$ for G_2 such that $B(\{w, z\}) \cap D''' \neq \phi$ where $z \in N(w)$, which is a contradiction as *G* is *TDDS.*

Case 2 $v \in \gamma_t(H)$

This case is similar to case 1.

Case 3 $u, v \in \gamma_t(H)$

If $u, v \in \gamma_t(H)$ then since *u* and *v* are bad vertices $\gamma_t(H) = \gamma_t(G_1) + \gamma_t(G_2)$. It is not possible that $\gamma_t(H) < \gamma_t(G_1) + \gamma_t(G_2)$ this implies that $\gamma_t(H) = \gamma_t(G_1) + \gamma_t(G_2)$. Consider $H_\bullet uv$. *uv* is a cut vertex in $H_\bullet uv$. If $\gamma_t(H_\bullet uv) < \gamma_t(H)$. Then $uv \in D$ where *D* is a $\gamma_t - set$ for $H_\bullet uv$. ie., either *u* is a good vertex w. r. to D_1 or *v* is a good vertex w. r. to D_2 is not possible. $\gamma_t(H_\bullet uv) = \gamma_t(H)$ also $\gamma_t(H_\bullet uv) = \gamma_t(H)\ \forall\ u, v \in V(G_1)$ or $V(G_2)$. Hence *H* is *TDDS.* □

Theorem 3.6

Let G_1 and G_2 be TDDS graph. Let $u \in V(G_1)$ and $v \in V(G_2)$. Let H be a graph generated by merging the vertices u and v then H is TDDS if, both u and v are bad vertices

Proof

Let G_1 and G_2 be *TDDS* graph. Let $u \in V(G_1)$ and $v \in V(G_2)$ be a bad vertices and D_1 and D_2 be a dominating sets for G_1 and G_2. H is obtained by merging the vertices u and v. uv cannot be included in $\gamma_t(H)$. If $uv \in \gamma_t(H)$, then since uv is a cut vertex, there is a γ_t - *set* for G_1 and G_2 including u and v a contradiction as u and v are bad vertices ie., uv is a bad vertex for graph H also. $\gamma_t(H) = \gamma_t(G_1) + \gamma_t(G_2)$. Also $\gamma_t(H \bullet uv) = \gamma_t(H) \forall u, v \in V(G_1)$ or $V(G_2)$. Hence H is *TDDS*.

Conclusion

We have obtained some methods of generating new graphs from existing ones. Can we characterize trees that are total domination dot stable graphs by using graph operations that have been used here?

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