THE MATHEMATICAL DESCRIPTION OF THE MOBILITY MECHANISMS USING LIE SPACE AND MATRIX GROUP SE(3)

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Abstract.

This paper proposes an approach to the structural synthesis of parallel robots based on the Lie algebras of the displacement groups. This approach allows the automation of the synthesis and eliminates the disadvantages of both methods based on the using of displacement subgroups, and methods based on the screw theory. Parallel robot is presented consisting of the moving platform and the stationary base connected via several serial kinematic chains. It is shown that movement of kinematic chain correspond to Lie subgroups of displacement, called dis-placement subgroups generators. Structural synthesis was performed for the case in which the mobile platform speed relative to the fixed basement forms a subalgebra of the Lie algebra of the displacement group of Euclidean space. In this case, we obtain three types of parallel robots. We also consider the case when the speed of mobile platform relative to the fixed basement forms a subspace, which is not a subalgebra of the displacement group of Euclidean space. It is divided into two types of robots.

Key words: Parallel robot, Lie group, Lie algebra, kinematic chain, displacement group.

Introduction.

The problem of structural synthesis of multi-loop spatial mechanisms is to find all structures (architectures) multiple-spatial mechanisms for a predetermined number of degrees of freedom. The problem of structural synthesis of multi-loop spatial mechanisms originated in the late 1960s and was probably the least popular area of research in mechanics for several decades. However, in the early 1990s in this area there has been some progress [1-2].
Structural synthesis of multiple-spatial arrangements in which all paths have the same number of constraint equations, considered by some authors (e.g., [3]), based on the use of mobility criterion. In 1994 was made [4, 5] structural synthesis of spatial mechanisms containing prismatic and rotational kinematic pairs in which all the circuits have 6 independent conditions of communication. Spatial arrangements were synthesized passive prismatic kinematic pairs. As mentioned above, the next step of structural synthesis of parallel mechanisms is the choice of the drive kinematic pairs. However, this problem for a longtime been considered. One reason for this is that the majority of the authors engaged in the synthesis of spatial mechanisms consisting of kinematic chains with one degree of freedom. For such kinematic chain drive can be any kinematic pair. Another reason is that the condition if the proper drive kinematic pairs has been formulated in [6] as follows: "For the F-movable mechanism set out the drive kinematic pairs is true if the number of degrees of freedom kinematic chain obtained by blocking all drive kinematic pairs is 0". However, when choosing a drive kinematic pairs with this condition there is a very difficult task of determining the number of degrees of freedom of the mechanism. As stated in [7], sets the drive for the two pairs of kinematic PM is not operational. For spatial mechanisms containing translational and rotational kinematic pairs in which each circuit has six independent constraint equations, it proposed a new criterion of checking of the selection drive kinematic pairs. Currently, the problem as a structural synthesis of parallel mechanisms and select the drive for kinematic pairs of spatial kinematic chains is not yet fully resolved.

1. Structural synthesis of parallel mechanisms with a predetermined movement path.

Due to a large variety of applications of parallel mechanisms, the trajectory required by these movements may vary significantly. There is an urgent need to find new structures parallel mechanisms executed by the specified movement trajectory. Additionally, the new structure of parallel mechanisms needed in connection with the development of hybrid kinematics machines in which two parallel mechanisms working together. Since the trajectory of the movable platform contains more information than the number of its degrees of freedom, and because of the need for parallel arrangements with a number of degrees of freedom is less than 6, structural synthesis of parallel mechanisms with a given movement trajectory since the 1990s is receiving increasing attention [8-11]. There is a view that parallel mechanisms with 6 degrees of freedom are applicable in all cases, and therefore questioned the need for structural synthesis of parallel mechanisms with many degrees of freedom, less 6. One of the main causes of structural synthesis of parallel mechanisms with many degrees of freedom, at least 6. It is to reduce the cost of manufacture. Another reason is that, in general, reducing the number of degrees of freedom increases the
Approaches to structural synthesis of parallel mechanisms with a given movement trajectory can be divided into four classes: (1) an approach based on the screw theory, (2) an approach based on the theory of groups of motions (see, eg, (3) an approach based on open chains with rigid end and (4) an approach based on virtual circuits.

2. Lie groups and movements subgroups.

A set of \( \{D\} \) representing a rigid body, has the structure of a particular group, the group of displacements. Such a group uniquely corresponds to the special Euclidean matrix group \( \text{SE}(3) \) Which is defined as the set of matrices of the form

\[
\text{SE}(3) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}
\]

where \( R \) — Rotation matrix, \( p \) — 3-dimensional vector, \( \text{SE}(3) \) — Continuous group, and any open set of elements \( \text{SE}(3) \) uniquely corresponds to an open set in \( \mathbb{R}^6 \). In mathematical terms, \( \text{SE}(3) \) — Differentiable manifold, called a Lie group. There are subgroups of movements, which play an important role in the structural synthesis. An exhaustive list of subgroups Lee gave Herve [12].

The consistent position of the two elements of the subgroups, called composition can form an element of the other subgroups. For example, the combination of three elements \( \{T(u)\} \) with vectors \( u_1, u_2 \) and \( u_3 \) may form an element of translational motion of the subgroup \( \{T\} \) in space while satisfying certain conditions vectors \( u \).

Intersection operation obtained when subjected to one of several solid elements subgroups is more important. Herve [12] gives rules that are subject to this operation (for example, the intersection of two elements \( \{D\} \) constitutes an element of the \( \{D\} \). An interesting case of intersection - the intersection of two elements of the subgroups \( \{X(w)\} \) and \( \{X(w)\} \) if \( w \neq w' \) gives subgroup element \( \{T\} \).

Movements of Lie subgroups generated by kinematic chains, called is generators movements subgroups.

Synthesis of parallel mechanisms based on the theory of groups is based on the following basic principles:

- Determining which subgroup it must have a working body to provide a predetermined number of degrees of freedom;
- Definition of all possible subgroups and a corresponding kinematic chain of which may consist of a robot support, the intersection of which gives a subgroup;
- Identification of all movements of generators for these subgroups: it is kinematic chains of the robot.
3. An approach based on the theory of screws.

An approach based on the theory of groups, led to the discovery of many new possible structures. However, the group \(D\) has specific properties that are not reflected in the structure of its Lie group. Correspondingly, an approach based on the theory of groups is difficult to apply to specific cases of mobility, such as paradoxical mechanisms.

We can extend the concept of the group, considering the tangent space at the identity element, which is a vector space, called the Lie algebra of a Lie group. Because we start from \(SE(3)\), appropriate Lie algebra \(se(3)\) is a vector space of instantaneous velocity, whose elements - the 6-dimensional vectors of the form \((W, v)\), where \(W\) — angular velocity of a rigid body, and \(v\) — its linear speed. These elements are called the kinematic screws.

Over the past 60 years, the theory of screws and its applications to mechanics developed Dimentberg, Glazunov, Young and Freydenshtayn, Uolrdon, Roth, Hunt, Phillips, Duffy, Angeles [13-16].

Important for movement forces and torques can be represented as a pair of three-dimensional vectors \((F, M)\) called power screw. Power and kinematic screw called reciprocal if

\[W \cdot M + v \cdot F = 0\]

When connecting the kinematic chain with a solid core idea is that the ability to instantly kinematic screws reciprocal power screws, cash kinematic chain (called reaction force screws). In other words, the number of degrees of freedom of a rigid body is determined by reaction force screws. For parallel robots can be argued as follows:

- The kinematic screw of the mobile platform is an intersection of the kinematic screw of the robot supports;
- Power screw the mobile platform is the union of all power screws robot supports. On the basis of the above statements the following synthesis methodology has been developed:
  - To find a group of power screws mutually desired kinematic screw of the mobile platform;
  - Determine the power screws robot kinematic chains whose union gives;
  - Identify all possible structures of kinematic chains, generating power by appropriate screws;
  - Regarded as the power and kinematic screws instantaneous (synthesis therefore, based on the theory of screws called polhodom first order), you must check is not instantaneous (full) mobility platform.

The disadvantage of this approach is the difficulty of automation, especially steps 3 and 4. It has the advantage of group theory: after calculation a possible restrictions in step 2, step 3 motion generators are easily identified. On the other side, by using an approach based on the theory of groups, the structure can be skipped, which are not within the
4. Synthesis of series circuits generating subset $\text{SE}(3)$

Consider the elements of group theory in relation to the group of Euclidean motions of $\text{SE}(3)$.

In the group of motions of a Euclidean space $\text{SE}(3)$ subgroup can be identified:

- $R_{P,u_1}$ — Rotation around the point $P$, positioned on the axis $u_1$, in a plane perpendicular to this axis;
- $T_{u_1}$ — Linear motion along the axis $u_1$;
- $H_{P,u_1,p}$ — Screw displacement of point $P$ along the axis $u_1$, with parameter $p$;
- $T_{u_1,k}$ — Linear displacement in a plane perpendicular to the axis $u_1$;
- $C_{P,u_1}$ — Rotation around the point $P$, positioned on the axis $u_1$, in a plane perpendicular to the axis and the linear movement along said axis;
- $T_{u_1,k_1,k_2;k_3}$ — Linear motion along three mutually orthogonal axes;
- $G_{u_1}$ — Linear motion in a plane perpendicular to the axis $u_1$ and rotating around this axis;
- $S_0$ — Arbitrary rotation in three-dimensional space about a fixed center $O$;
- $Y_{u_1,p}$ — Linear displacement in a plane perpendicular to the axis $u_1$, and a helical movement along this axis with the parameter $p$;
- $X_{u_1}$ — Schoenflies movement.

Consider a pair of subgroups $\text{SE}(3)$, which are subsets of the composition $\text{SE}(3)$. All possible to reduce the composition of the two tables, the first of which shows a pair of forming subsets $\text{SE}(3)$ and not containing a screw kinematic pair, and the second containing kinematic screw pair.

**Table 1: Classification of sub-groups of motions of Euclidean space not containing a screw kinematic pair, the composition of which does not form a subgroup.**

<table>
<thead>
<tr>
<th>№</th>
<th>Subgroups</th>
<th>The composition of subgroups and its dimension</th>
<th>The subset and its dimension</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$R_{P,u_1} \cdot R_{Q,u_2}$</td>
<td>$\text{SE}(3)$ 6</td>
<td>$T_2 \times R_2$ 2</td>
<td>A.d</td>
</tr>
<tr>
<td>2</td>
<td>$R_{P,u_1} \cdot R_{Q,u_2}$</td>
<td>$G_\nu$ 3</td>
<td>$T_2 \times R_1$ 2</td>
<td>d</td>
</tr>
<tr>
<td>3</td>
<td>$R_{P,u_1} \cdot R_{Q,u_2}$</td>
<td>$S_\nu$ 3</td>
<td>$T_2 \times R_2$ 2</td>
<td>c</td>
</tr>
<tr>
<td>4</td>
<td>$R_{P,u_1} \cdot T_{u_2}$</td>
<td>$X_\nu$ 4</td>
<td>$T_2 \times R_1$ 2</td>
<td>a</td>
</tr>
</tbody>
</table>
Table 2: Classification of sub-groups of motions of Euclidean space containing the screw kinematic pair, the composition of which does not form a subgroup.

<table>
<thead>
<tr>
<th>№</th>
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<th>The composition of subgroups and its dimension</th>
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<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( R_{p,u_1} \cdot H_{Q_{u_2,p}} )</td>
<td>SE(3) 6</td>
<td>( T_2 \times R_2 ) 2</td>
<td>A,c,d</td>
</tr>
<tr>
<td>2</td>
<td>( R_{p,u_1} \cdot H_{Q_{u_2,p}} )</td>
<td>( X_v ) 4</td>
<td>( T_2 \times R_1 ) 2</td>
<td>b</td>
</tr>
<tr>
<td>3</td>
<td>( R_{p,u_1} \cdot Y_{u_2,p} )</td>
<td>SE(3) 6</td>
<td>( T_3 \times R_2 ) 4</td>
<td>A,b</td>
</tr>
<tr>
<td>4</td>
<td>( T_{u_1} \cdot H_{Q_{u_2,p}} )</td>
<td>( X_v ) 4</td>
<td>( T_2 \times R_1 ) 2</td>
<td>A</td>
</tr>
<tr>
<td>5</td>
<td>( T_{u_1} \cdot H_{Q_{u_2,p}} )</td>
<td>( Y_{v,p} ) 3</td>
<td>( T_2 \times R_1 ) 2</td>
<td>D</td>
</tr>
<tr>
<td>6</td>
<td>( H_{p,u_1,p} \cdot H_{Q_{u_2,p}} )</td>
<td>SE(3) 6</td>
<td>( T_2 \times R_2 ) 2</td>
<td>A,c,d</td>
</tr>
<tr>
<td>7</td>
<td>( H_{p,u_1,p} \cdot H_{Q_{u_2,p}} )</td>
<td>( Y_{v,p} ) 3</td>
<td>( T_2 \times R_1 ) 2</td>
<td>b</td>
</tr>
<tr>
<td>8</td>
<td>( H_{p,u_1,p} \cdot H_{Q_{u_2,p}} )</td>
<td>SE(3) 6</td>
<td>( T_2 \times R_2 ) 2</td>
<td>A,c,d</td>
</tr>
<tr>
<td>9</td>
<td>( H_{p,u_1,p} \cdot H_{Q_{u_2,p}} )</td>
<td>( X_v ) 4</td>
<td>( T_2 \times R_1 ) 3</td>
<td>b</td>
</tr>
<tr>
<td>10</td>
<td>( H_{p,u_1,p} \cdot C_{Q_{u_2}} )</td>
<td>SE(3) 6</td>
<td>( T_2 \times R_1 ) 2</td>
<td>A,c,d</td>
</tr>
<tr>
<td>11</td>
<td>( H_{p,u_1,p} \cdot C_{Q_{u_2}} )</td>
<td>( X_v ) 4</td>
<td>( T_3 \times R_1 ) 3</td>
<td>b</td>
</tr>
<tr>
<td>12</td>
<td>( H_{p,u_1,p} \cdot T_{u_2} )</td>
<td>( X_v ) 4</td>
<td>( T_3 \times R_1 ) 3</td>
<td>A,d</td>
</tr>
</tbody>
</table>
13. $H_{P_u,\alpha_4,\rho} \cdot S_O$  
14. $H_{P_u,\alpha_4,\rho} \cdot G_{\nu_4}$  
15. $H_{P_u,\alpha_4,\rho} \cdot Y_{\nu_4,\rho}$  
16. $H_{P_u,\alpha_4,\rho} \cdot X_{\nu_4}$  
17. $C_{P_u,\alpha_4,\rho} \cdot Y_{\nu_4,\rho}$  
18. $C_{P_u,\alpha_4,\rho} \cdot X_{\nu_4,\rho}$  
19. $G_{\nu_4,\rho} \cdot Y_{\nu_4,\rho}$  
20. $Y_{\nu_4,\rho} \cdot Y_{\nu_4,\rho}$  
21. $Y_{\nu_4,\rho} \cdot X_{\nu_4,\rho}$  
22. $Y_{\nu_4,\rho} \cdot X_{\nu_4,\rho}$  
23. $Y_{\nu_4,\rho} \cdot X_{\nu_4,\rho}$

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>Unit vectors defining the subset associated with each direction, not parallel and not perpendicular. If the associated axis, these axes do not intersect with each of the subgroups.</td>
</tr>
<tr>
<td>14</td>
<td>Unit vectors defining the subset associated with each direction of movement parallel. If each of the subgroups associated axis, these axes are coaxial.</td>
</tr>
<tr>
<td>15</td>
<td>The axes associated with subgroups intersect but are not parallel and not perpendicular.</td>
</tr>
<tr>
<td>16</td>
<td>The unit vectors defining associated with each subgroup direction, perpendicular. If each sub-direction, perpendicular. If the associated axis, these axes do not intersect with each of the subgroups.</td>
</tr>
<tr>
<td>17</td>
<td>Axis associated with one of the subgroups, does not pass through the point associated with the other subgroup.</td>
</tr>
<tr>
<td>18</td>
<td>Axis associated with one of the subgroups, is passes through the point associated with the other subgroup.</td>
</tr>
<tr>
<td>19</td>
<td>There are no conditions.</td>
</tr>
<tr>
<td>20</td>
<td>Points associated with each of the subgroups is varied.</td>
</tr>
</tbody>
</table>

As an example, consider a motion corresponding to the composition of subgroups $T_{u_1} \cdot S_O$ and $S_O \cdot T_{u_1}$, We obtain kinematic chain corresponding to this composition (Figure 1).

Figure 1. Serial kinematic chain generating group composition $T_{u_1} \cdot S_O$ and $S_O \cdot T_{u_1}$
Let us consider further structural synthesis of parallel M1-, M2-, M3-type mechanisms. Synthesis M1 type mechanisms discussed in [17].

5. Synthesis of parallel mechanisms M2, M3

Consider a parallel mechanism consisting of the movable platform \(m\) and the stationary base \(b\), connected by \(k\) serial kinematic chain (Figure 2).

![Parallel Mechanism](image)

**Figure 2. Parallel Mechanism.**

When the structural synthesis of parallel mechanisms, two cases can be distinguished:

1. Structural synthesis of parallel mechanisms when the movable platform velocity relative to the stationary base forms subalgebra of Lie algebra \(se(3)\) of Euclidean space \(SE(3)\) movements groups. The this case can distinguish three type parallel mechanisms:
   - Type A1: Required subalgebra \(A_R \subset se(3)\) is the intersection of subalgebras generated by a series of connected chains and all these subalgebras are equivalent.
   - Type A2: Required subalgebra \(A_R \subset se(3)\) is the intersection of subalgebras generated by a series of connected circuits, and not all of these subalgebras are equivalent.
   - Type A3: Required subalgebra \(A_R \subset se(3)\) belongs to (and may be equivalent to) the intersection of the spaces generated by a series of connected chains, some of which, but not all, are subalgebras.

2. Structural synthesis of parallel mechanisms when the movable platform velocity relative to the stationary base forming subspace non subalgebra of Lie algebra \(se(3)\) of Euclidean space \(SE(3)\) groups movements. The this case can distinguish two type parallel mechanisms:
   - Type S1: Required space \(V_R \subset se(3)\) is the intersection of subspaces generated by a series of connected chains and all these subspaces are equivalent.
Type S2: Required space \( V_R \subseteq se(3) \) is the intersection of subspaces generated by a series of connected circuits, and not all of these are equivalent to the subspace.

Consider the parallel synthesis of M3-type mechanisms.

Let \( A_R \subseteq se(3) \) – required subalgebra for a predetermined movement. Then

\[
A_R = \bigcap_{j=1}^{k} A_j^{m/b}, \text{ respectively, } A_a^{m/b} \subseteq A_j^{m/b}, j=1,2,\ldots,k
\]

In this case, the required subalgebra is not equivalent to all the subalgebras generated serial kinematic chains. However, all these subalgebras must contain the required subalgebra, i.e.

\[
A_R \subseteq A_j^{m/b}, j = 1,2,\ldots,k
\]

And the intersection of subalgebras is

\[
A_a^{m/b} = \bigcap_{j=1}^{3} A_j^{m/b} = \begin{bmatrix}
1/2 \sqrt{3} & 1/2 \sqrt{3} & 1/2 \sqrt{3} & 0 \\
1/2 & 1/2 & 1/2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 5 & 10 & 1/2 \sqrt{3} \\
0 & -5 \sqrt{3} & -10 \sqrt{3} & 1/2 \\
5 & 20 & 3 & 0
\end{bmatrix}
\]

This subalgebra corresponds to the subalgebra spatial movements \( t_{u_1,u_2,u_3} \). The result is a mechanism called Delta (Figure...
Consider the parallel synthesis of M3-type mechanisms.

Let $A_R \subset se(3)$ – required subalgebra for a predetermined movement. Then

$$A_R = V_{a}^{mb} = \bigcap_{j=1}^{k} V_{j}^{mb},$$

respectively, $A_a^{mb} \subset V_j^{mb}, j=1,2...,k$

As an example, consider a mechanism for carrying out the Schoenflies movement. Subspaces generated by serial kinematic chains, represents the space of Jacobians columns:

$$V_{1}^{mb} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -3 & -6 & -9 & 5 \\
1 & 0 & 0 & 0 & -10 \\
0 & 0 & 2 & 6 & 10
\end{bmatrix}$$

$$V_{2}^{mb} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 4 & 8 & -9 \\
0 & 0 & 3 & -2 & 0 \\
1 & 0 & 0 & 0 & 6
\end{bmatrix}$$

$$V_{3}^{mb} = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 7 \\
0 & 2 & 5 & 9 & -4 \\
0 & 0 & 2 & -3 & 0
\end{bmatrix}$$

and the intersection of subalgebras is

$$A_a^{mb} = V_a^{mb} = \bigcap_{j=1}^{3} V_j^{mb} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
28 & 16 & -16 & 16 \\
-7 & -4 & 0 & -4 \\
3 & 5 & 0 & 0
\end{bmatrix},$$

and corresponds to the Schoenfliesubalgebra $\chi_{uu}$. The resulting mechanism is shown in Figure 4.
**Conclusion.** In conclusion, it should be noted that the analyzed approaches to structural synthesis of parallel mechanisms, based on the theory of screws and movements of group theory have significant drawbacks. The method, based on the theory of groups of motions is difficult to apply to some specific cases of mobility, using a method based on the theory of screws, it is necessary to check the instantaneous mobility platform. The proposed method of structural synthesis of parallel mechanisms, based on an analysis of subalgebras and the Lie algebra subspace, eliminates these drawbacks. An example of structural synthesis of parallel mechanisms for the case, if the mobile platform speed relative to the fixed base forms a subalgebra of the Lie algebra se (3) of the Euclidean space group, confirming the effectiveness of this approach.

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**References**


