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QUADRATURE FORMULAS FOR CALCULATING THE HADAMARD INTEGRAL OF A SPECIAL FORM

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Abstract

Currently, the issues related to finding solutions to some economic problems, such as problems pertaining to the queuing systems are of great interest. These problems, in turn, result in, inter alia, the need to calculate the Hadamard integrals of a special form.

The study includes the construction of an optimal quadrature formula for the approximate solution to the Hadamard integral of a special form; such formula is selected depending on the singularities of integrals. This is because of the fact that an error of the quadrature formula for a hypersingular integral is a point function of any singularity. There is a movable singularity, when parameters of the quadrature formula do not depend on the singularity's position, and a fixed singularity, when the quadrature formula nodes are the singularity's point functions. In this way, choice of the quadrature formula nodes depends on the choice of singularities. Assessment of the constructed quadrature formula's error is established in the considered density class, and the order of formulas with arbitrary multiplicity nodes is optimized. The main results and conclusions are given; they were obtained for quadrature formulas of hypersingular integrals with the singularities, both movable and fixed, which can be used for solving boundary value problems, especially in simulating the fractional (or fractal) dynamic processes.

Key words: Quadrature formulas, Hadamard integrals, optimization.

Introduction

The quadrature formulas relate to the most commonly used apparatus for approximate calculation of integrals; studies in this field were carried out quite actively. It is important for researchers to construct such quadrature formulas that would ensure the greatest accuracy in calculating the integrals. In this connection, the question arises about the optimization of quadrature formulas for various integrals. So, for example, in works [1-6, 7], quadrature formulas for singular integrals, integral equations with special kernels were studied.

At present, equations with fractional integral operators are studied very actively. There is a number of theoretical and practical problems, which lead to the need to solve equations with fractional integration operators [8-13]. In recent years, the works, in which numerical methods for some classes of equations [14-18] are proposed, appear in the scientific literature. But the question remains as to optimization of quadrature formulas for equations with Hadamard integrals. The study includes the construction of an optimal quadrature formula for the approximate solution to the Hadamard integral of a special form; such formula is selected depending on the singularities of integrals.

Methods. Statement of problem.

In class $W^rH^\alpha[-1;+1]$ of the functions *f(x) r*-times continuously differentiable on the interval [-1;+1], which satisfy the condition

$\omega(f^{(r)},\delta) = \sup |f^{(r)}(x') - f^{(r)}(x")| \le \omega(\delta)$ at $|x' - x"| \le \delta, x', x \in [-1;1]$,

where $\omega(\delta), 0 \le \delta \le 2 -$ is a given upward-convex modulus of continuity, let us consider the singular integral

$$S(m,\alpha,f,t) = \int_{-1}^{1} \frac{f(x)}{(x-t)^m ln^\alpha|x-t|} dx, \qquad -1 < t < 1, 0 < \alpha \le 1, \qquad (1)$$

($m \ge 2 -$ is an integer), which is understood in the sense of the finite Hadamard value:

$$S(m,\alpha,f,t) = \lim_{\varepsilon \to +0} \left(\int_{-1}^{t-\varepsilon} \frac{f_1(x)}{(x-t)^m} dx + \int_{t+\varepsilon}^{1} \frac{f_1(x)}{(x-t)^m} dx - \frac{\varphi(t)}{\varepsilon^{m-1}} \right),$$

where $\varphi(t) = \sum_{k=0}^{m-2} \frac{f_1^{(k)}(t)}{k!} \cdot \frac{\varepsilon^k(1+(-1)^{m-k})}{m-k-1}$, $f_1(t) = \frac{f(t)}{ln^\alpha|x-t|}$.

Let us assume that the below quadrature formula corresponds to the integral (1):

$$S(m,\alpha,f,t) = \sum_{i=0}^{N} \sum_{j=0}^{\gamma_i} a_{ij}(t) f^{(j)}(x_i) + R(m,\alpha,t,X_N,A_N,\Gamma_N), \qquad -1 < \qquad (2)$$

such formula is determined by the matrix of weight functions $A_N = \{a_{ij}(t)\}$, independent of *f*, node vectors $X_N = \{x_i\}, -1 \le x_0 < x_1 < \cdots < x_N \le 1$, by the integer vector $\Gamma_N = \{\gamma_i\}, 0 \le \gamma_i \le r + m - 1, i = \overline{0,N}$, and the one that gives multiplicities of the quadrature formula nodes.

It should be noted that the remainder $R(m,\alpha,t,X_N,A_N,\Gamma_N)$ of the quadrature formula (2) is a point function of $t\in[-1,1]$, that is why two optimal quadrature formula problems can be stated for singular integrals (1).

Problem 1. Suppose that the parameters of the quadrature formula (2) are selected independently from the position of the point $t\in[a,b]\subset[-1,1]$. Then to estimate an error related to the quadrature formula of form (2) in the class $W^{r+m-1}H_\omega[-1;1]$ on the interval $[a,b]$ at the fixed set Γ_N in an optimal way, specify the formula:

$$\epsilon_N(m,\alpha,W^{r+m-1}H_\omega[-1;1],[a,b],\Gamma_N)=inf\,sup\,sup|R(m,\alpha,t,X_N,A_N,\Gamma_N)|,$$

here *inf* is taken according to $X_N,A_N,$ and *sup* – according to functions $f(t)\in W^{r+m-1}H_\omega[-1;1],t\in[a,b].$

The order optimal quadrature formula that is defined by the nodes x_N^0 and weight matrix A_N^0 is the one, for which it is performed as follows:

$$\epsilon_N(m,\alpha,W^{r+m-1}H_\omega[-1;1],[a,b],\Gamma_N)\asymp sup\,sup|R(m,\alpha,t,X_N^0,A_N^0,\Gamma_N)|.$$

Problem 2. Let $t\in[a,b]\subset[-1,1]-$ be a fixed point. Then the optimal estimate of an error related to the quadrature formula of form (2) in the class $W^{r+m-1}H_\omega[-1;1]$ on the interval $[a,b]$ at the fixed set Γ_N will be given by the formula:

$$\epsilon_N(m,\alpha,W^{r+m-1}H_\omega[-1;1],[a,b],\Gamma_N)=inf\,sup|R(m,\alpha,t,X_N,A_N,\Gamma_N)|,$$

where *inf* is taken according to $X_N,A_N,$ and *sup* – according to functions $f\in W^{r+m-1}H_\omega[-1;1].$

In this case the order optimal quadrature formula will be the formula that is determined by the nodes x_N^0 and weight matrix A_N^0 and that satisfies the condition:

$$\epsilon_N(m,\alpha,W^{r+m-1}H_\omega[-1;1],[a,b],\Gamma_N)\asymp sup|R(m,\alpha,t,X_N^0,A_N^0,\Gamma_N)|.$$

It is clear that in this case the nodes and weights are functions of the point *t.*

Results

Hypersingular integrals and equations with them on an interval

Consider the integral

$$J=I(x_0)=\int_a^b\frac{dx}{(x_0-x)^2ln\left|\frac{x_0-x}{c}\right|},\qquad x_0\in(a,b),c>b-a. \tag{3}$$

The integral $I(x_0)$ will be understood in the sense of Hadamard, i.e.

$$J = \lim_{\varepsilon \to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{dx}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|} + \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{dx}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|}\right) =$$

$$= \lim_{\varepsilon \to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{dx}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|} + \frac{2}{c} Li\left(\frac{c}{\varepsilon}\right)\right) = \frac{1}{c} Li\left(\frac{c}{x_0-a}\right) - \frac{1}{c} Li\left(\frac{c}{x_0-b}\right), \qquad (4)$$

where the function $Li(x) = \int_0^x \frac{dt}{\ln |t|}$, $I = [a,b]$, $O_\varepsilon(x_0) = (x_0-\varepsilon, x_0+\varepsilon)$,

$I\setminus O_\varepsilon(x_0) = [a, x_0-\varepsilon) \cup (x_0+\varepsilon, b]$. The integral *J* in the formula (4) is called a hypersingular integral on an interval.

Further, we should consider an integral of the form:

$$I(x_0) = \int_a^b \frac{g(x)dx}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|}, \qquad x_0 \in (a,b), c > b-a, \qquad (5)$$

where function $g(x)$ is specified on the interval $[a,b]$.

We will understand the integral (5) in the sense of an equality:

$$I(x_0) = \lim_{\varepsilon \to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{g(x)dx}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|} + \frac{2}{c} Li\left(\frac{c}{\varepsilon}\right)\right). \qquad (6)$$

The following theorems are correct for the integral (5).

Теорема 1. The integral (5) exists for any function $g(x) \in H_1(\alpha)$ on the interval $[a,b]$, where $H_1(\alpha) -$ is a Holder class of the degree α for the function $g'(x)$, i.e. $g'(x) \in H(\alpha)$.

Proof.

$$I(x_0) = \lim_{\varepsilon \to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{g(x)-g(x_0)-g'(x_0)(x-x_0)}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|} dx\right) +$$

$$+\lim_{\varepsilon \to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{g(x_0)dx}{(x_0-x)^2 \ln\left|\frac{x_0-x}{c}\right|} - \int_{I\setminus O_\varepsilon(x_0)} \frac{g'(x_0)dx}{(x_0-x) \ln\left|\frac{x_0-x}{c}\right|} + \frac{2g(x_0)}{c} Li\left(\frac{c}{\varepsilon}\right)\right)$$

$$=$$

$$= \lim_{\varepsilon\to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{g(x)-g(x_0)-g'(x_0)(x-x_0)}{(x_0-x)^2 ln\left|\frac{x_0-x}{c}\right|}dx\right)+$$

$$+g(x_0)\lim_{\varepsilon\to 0}\left(\int_{I\setminus O_\varepsilon(x_0)} \frac{dx}{(x_0-x)^2 ln\left|\frac{x_0-x}{c}\right|}+\frac{2}{c}Li\left(\frac{c}{\varepsilon}\right)\right)-$$

$$-g'(x_0)\lim_{\varepsilon\to 0}\int_{I\setminus O_\varepsilon(x_0)} \frac{dx}{(x_0-x) ln\left|\frac{x_0-x}{c}\right|}= \quad (7)$$

$$= I_1(x_0)+g(x_0)I_2(x_0)-g'(x_0)I_3(x_0).$$

In the formula (7) the integral $I_1(x_0)$ exists as an improper, because $g'(x)\in H(\alpha)$ and, consequently,

$$|g(x)-g(x_0)-g'(x_0)(x-x_0)|\le A|x-x_0|^{1+\alpha}.$$

The integral $I_2(x_0)$ exists in the sense of a definition (4). The integral $I_3(x_0)$ exists in the sense of the Cauchy principal value.

$$\lim_{\varepsilon\to +0}\left(\int_a^{x_0-\varepsilon}\frac{dx}{(x_0-x)ln\left|\frac{x_0-x}{c}\right|}+\int_{x_0+\varepsilon}^{b}\frac{dx}{(x_0-x)ln\left|\frac{x_0-x}{c}\right|}\right)=$$

$$=\frac{1}{c}ln\left|ln\left|\frac{x_0-a}{c}\right|\right|-\frac{1}{c}ln\left|ln\left|\frac{x_0-b}{c}\right|\right|.$$

Thus, Theorem 1 is completely proved.

Theorem 2. Suppose that the function $g(x)\in H_1(\alpha)$ is on the interval $[a,b]$. Then the following partial integration formula is correct for the integral (5):

$$\int_a^b \frac{g(x)dx}{(x_0-x)^2 ln\left|\frac{x_0-x}{c}\right|}=\frac{-g(b)}{c}Li\left(\frac{c}{x_0-b}\right)+\frac{g(a)}{c}Li\left(\frac{c}{x_0-a}\right)+$$

$$+\frac{1}{c}\int_a^b g'(x)Li\left(\frac{c}{x_0-x}\right)dx. \quad (8)$$

Proof. In integrating by parts, we obtain in the formula (6):

$$I(x_0)=\lim_{\varepsilon\to +0}\left\{\frac{-g(x)}{c}Li\left(\frac{c}{x_0-b}\right)\Big|_a^{x_0-\varepsilon}-\frac{g(x)}{c}Li\left(\frac{c}{x_0-b}\right)\Big|_{x_0+\varepsilon}^{b}+\right.$$

$$+\frac{1}{c}\int_{I\setminus O_{\varepsilon}(x_0)} g'(x)Li\left(\frac{c}{x_0-x}\right)dx+\frac{2g(x_0)}{c}Li\left(\frac{c}{\varepsilon}\right)\Bigg\}=$$

$$=\frac{-g(b)}{c}Li\left(\frac{c}{x_0-b}\right)+\frac{g(a)}{c}Li\left(\frac{c}{x_0-a}\right)+\lim_{\varepsilon\to+0}\left\{\frac{-g(x_0+\varepsilon)}{c}Li\left(\frac{c}{\varepsilon}\right)-\right.$$

$$\left.-\frac{-g(x_0-\varepsilon)}{c}Li\left(\frac{c}{\varepsilon}\right)+\frac{2g(x_0)}{c}Li\left(\frac{c}{\varepsilon}\right)\right\}+\frac{1}{c}\lim_{\varepsilon\to+0}\left(\int_{I\setminus O_{\varepsilon}(x_0)} g'(x)Li\left(\frac{c}{x_0-x}\right)dx\right)$$

$$=\frac{-g(b)}{c}Li\left(\frac{c}{x_0-b}\right)+\frac{g(a)}{c}Li\left(\frac{c}{x_0-a}\right)+I_4(x_0)+I_5(x_0).$$

Since the integral $I_5(x_0)$ exists as an improper, and $g(x)\in H_1(\alpha)$, and it is fulfilled as follows:

$$|g(x_0+\varepsilon)+g(x_0-\varepsilon)-2g(x_0)|=|g(x_0+\varepsilon)-g(x_0)|-$$

$$-|g(x_0)-g(x_0-\varepsilon)|\le A\varepsilon^{1+\alpha},$$

therefore, the limit $I_4(x_0)$ is equal to zero for any values $x\in(a,b)$.

Thus, Theorem 2 is completely proved.

Note 1. If $g(a)=g(b)=0$ и $g(x)\in H_1(\alpha)$ on the interval $[a,b]$, then the formula (8) will be as follows:

$$\int_a^b \frac{g(x)dx}{(x_0-x)^2 ln\left|\frac{x_0-x}{c}\right|}=\frac{1}{c}\int_a^b g'(x)Li\left(\frac{c}{x_0-x}\right)dx. \qquad (9)$$

Note 2. The formula (9) is also correct, when $g(a)=g(b)=0$, a $g(x)\in H_1^*(\alpha)$ on the interval $[a,b]$, i.e. $g'(x)\in H^*(\alpha)$, it means:

$$g'(x)=\frac{\psi(x)}{(x-a)^{\nu}(b-x)^{\mu}},$$

where $\psi(x)\in H$ on the interval $[a,b]$, i.e. it is the Holder function on $[a,b]$ of a certain degree, and $\mu,\nu>1$.

If the function $g(x)$ satisfies the requirements of notes 1, 2, we say that $g(x)\in H_1^{(0,0),*}$ on the interval $[a,b]$.

Consider the following quadrature formula for a hypersingular integral (5) on the interval $[a,b]$. For this end, let us take, on the interval $[a,b]$, the points $x_k=a+(k-1)h$, $h=\frac{(b-a)}{n}$, $k=\overline{1,n+1}$, which divide the interval $[a,b]$ into n equal intervals $I_k=[x_k,x_{k+1}]$, $k=\overline{1,n}$, and the points $x_{0k}=x_k+\frac{h}{2}$, $k=\overline{1,n}$.

Let us say that the sets $E=\{x_k, k=\overline{1,n}\}$ and $E_0=\{x_{0k}, k=\overline{1,n}\}$ form a canonical dissection of the interval $[a,b]$. Replace the integral (5) at the point $x_{0j}\in E_0$ by the following sum:

$$S_n(x_{0j})=\sum_{k=1}^{n} g(x_{0k})\int_{I_k}\frac{dx}{(x_{0j}-x)^2 ln\left|\frac{x_{0j}-x}{c}\right|}=$$

$$=\frac{1}{c}\sum_{k=1}^{n} g(x_{0k})\left\{Li\left(\frac{c}{x_{0j}-x_{k+1}}\right)-Li\left(\frac{c}{x_{0j}-x_k}\right)\right\}.$$

Correct is the following

Theorem 3. Suppose that the function $g(x)\in H_1^*(\alpha)$ is on the interval $[a,b]$. Suppose also that the sets E and E_0 form the canonical partition of this interval. Then the following inequality is correct:

$$I_n(x_{0j})=|I(x_{0j})-S_n(x_{0j})|=$$
$$=\left|\frac{1}{c}\sum_{k=1}^{n} g(x_{0k})\left\{Li\left(\frac{c}{x_{0j}-x_{k+1}}\right)-Li\left(\frac{c}{x_{0j}-x_k}\right)\right\}-\right.$$
$$\left.-\int_a^b\frac{g(x)\,dx}{(x_{0j}-x)^2 ln\left|\frac{x_{0j}-x}{c}\right|}\right|\le\Theta(x_{0j}),\ j=\overline{1,n},$$

here the value $\Theta(x_{0j})$ satisfies the inequalities:

a) for all points $x_{0j}\in[a+\delta,b-\delta]$, where $\delta>0$ is an arbitrarily small number,

$$\Theta(x_{0j})\le C_\delta h^{\alpha_1+2},\qquad 0<\alpha_1<1, \tag{10}$$

b) for all points $x_{0j}\in[a,b]$

$$\sum_{j=1}^{n}\Theta(x_{0j})h\le Ch^{\alpha_2+2},\qquad 0<\alpha_2<1, \tag{11}$$

where C_δ, C — are some constants that do not depend on *n*.

Proof. Represent the integral $I_n(x_{0j})$ by the formula (8) and the sum $S_n(x_{0j})$ as:

$$S_n(x_{0j}) = -\frac{1}{c}\left\{-g(x_{0k})Li\left(\frac{c}{x_{0j}-a}\right) - \sum_{k=2}^{n}\frac{g(x_{0k})-g(x_{0k-1})}{h}\cdot h\cdot Li\left(\frac{c}{x_{0j}-x_k}\right) + + g(x_{0k})Li\left(\frac{c}{x_{0j}-b}\right)\right\}. \quad (12)$$

Then for $I_n(x_{0j})$ it can be written as follows:

$$I_n(x_{0j}) \le \left|\left(\frac{-g(b)}{c}+\frac{g(x_{0n})}{c}\right)Li\left(\frac{c}{x_{0j}-b}\right)\right| + \left|\left(\frac{-g(b)}{c}+\frac{g(x_{0n})}{c}\right)Li\left(\frac{c}{x_{0j}-b}\right)\right| + \left|\frac{1}{c}\int_a^b g'(x)Li\left(\frac{c}{x_{0j}-x}\right)dx - \frac{1}{c}\sum_{k=2}^{n}\frac{g(x_{0k})-g(x_{0k-1})}{h}\cdot h\cdot Li\left(\frac{c}{x_{0j}-x_k}\right)\right| =$$

$$= I_{1,n}(x_{0j}) + I_{2,n}(x_{0j}) + I_{3,n}(x_{0j}).$$

Let us estimate each term consecutively.

$$I_{1,n}(x_{0j}) = \left|\left(\frac{-g(b)}{c}+\frac{g(x_{0n})}{c}\right)Li\left(\frac{c}{x_{0j}-b}\right)\right| \le A\frac{1}{c}\left(\frac{h}{2}\right)^{\alpha}\int_0^b \frac{dx}{\ln\left|\frac{c}{x_{0j}-x}\right|} \le$$

$$\le \Theta_1(x_{0j}) \le \frac{A}{2c}\left(\frac{h}{2}\right)^{\alpha+2}.$$

Similarly, we find that $I_{2,n}(x_{0j}) \le \Theta_2(x_{0j}) \le \frac{A}{2c}\left(\frac{h}{2}\right)^{\alpha+2}$.

$$I_{3,n}(x_{0j}) = \frac{1}{c}\left|\int_a^b g'(x)Li\left(\frac{c}{x_{0j}-x}\right)dx - \sum_{k=2}^{n}\frac{g(x_{0k})-g(x_{0k-1})}{h}\cdot h\cdot Li\left(\frac{c}{x_{0j}-x_k}\right)\right| =$$

$$= \frac{1}{c}\left|\sum_{k=2}^{n}\int_{x_{k-1}}^{x_k} g'(x)Li\left(\frac{c}{x_{0j}-x}\right)dx - \frac{g(x_{0k})-g(x_{0k-1})}{h}\cdot h\cdot Li\left(\frac{c}{x_{0j}-x_k}\right)\right|.$$

Using the mean value theorem, we obtain:

$$I_{3,n}(x_{0j}) = \frac{1}{c}\left|\sum_{k=2}^{n} Li\left(\frac{c}{x_{0j}-x'}\right)\int_{x_{k-1}}^{x_k} g'(x)dx - \frac{g(x_{0k})-g(x_{0k-1})}{h}\cdot h \cdot Li\left(\frac{c}{x_{0j}-x_k}\right)\right|$$

$$\le \frac{1}{c}\left|\sum_{k=2}^{n} Li\left(\frac{c}{x_{0j}-x'}\right)[g(x_k)-g(x_{k-1})-g(x_{0k})+g(x_{0k-1})]\right|,$$

$x' \in (x_{k-1}, x_k)$.

Given the conditions, imposed on the function $g(x) \in H_1^*(\alpha)$ from the interval $[a,b]$, we obtain:

$$|g(x_k)-g(x_{0k})| \le A\left(\frac{h}{2}\right)^{\alpha}, \qquad |g(x_{k-1})-g(x_{0k-1})| \le A\left(\frac{h}{2}\right)^{\alpha}.$$

As a result, we have an estimate

$$I_{3,n}(x_{0j}) \le \Theta_3(x_{0j}) \le C_\delta(h)^{\alpha+2}, \qquad \alpha \in (0,1].$$

And each of the values $\Theta_p(x_{0j}), p = 1,2,3,$ satisfies the inequalities (10) and (11). Therefore, the sum of the values $\Theta_p(x_{0j}), p = 1,2,3,$ also satisfies these inequalities. Theorem 3 is completely proved.

Note. This implies that the quadrature sum $S_n(x_{0j})$ converges uniformly in points x_{0j} to the value of the integral $I(x_{0j})$, однако, but outside the fixed neighbourhood of endpoints of the interval $[a,b]$ and at the approach of the point x_{0j} to one of the endpoints, this error can grow indefinitely with the growth of *n*, but in general there is an integral convergence, i.e. the inequality (11) is satisfied.

Special case

Consider the equation

$$\int_{-1}^{1} \frac{g(x)dx}{(x_0-x)^2 ln\left|\frac{x_0-x}{c}\right|} = f(x_0), \quad x_0 \in (-1,1). \qquad (13)$$

Solution to the equation (13) will be sought in the class of functions $H_1^{(0,0),*}$ on the interval $[-1,1]$.. Hence, according to (9), the equation (13) is equivalent to the following equation:

$$\frac{1}{c}\int_a^b g'(x)Li\left(\frac{c}{x_0-x}\right)dx = f_1(x_0), \quad x_0 \in (-1,1), \;\; f(x_0) = cf_1(x_0)$$

if the following condition has been satisfied:

$$\int_{-1}^{1} g'(x)dx = g(1) - g(-1) = 0. \qquad (14)$$

Under the condition (14), the function below is the solution to the equivalent equation (13):

$$g'(x) = \frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{1-{x_0}^2}f(x_0)Li\left(\frac{c}{x_0-x}\right)dx_0,\ \ x \in [-1,1].$$

Consequently, the function below is the solution to the equation (13) in the class of functions $H_1^{(0,0),*}$ on the interval $[-1,1]$.

$$g(x) = \frac{1}{\pi^2}\int_{-1}^{1}\frac{d\tau}{\sqrt{1-\tau^2}}\int_{-1}^{1}\sqrt{1-{x_0}^2}f(x_0)Li\left(\frac{c}{\tau-x_0}\right)dx_0,\ \ x \in [-1,1].$$

Since at the successive integration, the order of integration can be rearranged, and, given that one integral is singular and the other is absolutely integrable, the desired function - exact solution to the equation (13) can be written as:

$$g(x) = \frac{1}{\pi^2}\int_{-1}^{1}\frac{1}{\sqrt{1-{x_0}^2}}\left(\int_{-1}^{1}\frac{1}{\sqrt{1-\tau^2}}Li\left(\frac{c}{\tau-x_0}\right)d\tau\right)f(x_0)dx_0, \qquad (15)$$

$$x \in [-1,1].$$

The numerical solution to the hypersingular integral equation will be searched based on the vortex pair method. For this purpose, we replace the equation (13) with a system of linear algebraic equations using canonical dissection of the interval $[-1,1]$.

$$\sum_{k=1}^{n} g_n(x_{0k})\left\{Li\left(\frac{c}{x_{0j}-x_{k+1}}\right) - Li\left(\frac{c}{x_{0j}-x_k}\right)\right\} = f(x_{0j}),\ \ j = \overline{1,n}. \qquad (16)$$

Correct is the following

Theorem 4. Suppose that the function $f(x_0)$ belongs to the class of functions *H* on the interval $[-1,1]$. Then, between the solution to the equation (16) and solution (15) to the equation (13) the following relation is satisfied:

$$|g(x_{0k}) - g_n(x_{0k})| \le Ch^{\lambda},\ \ \ k = \overline{1,n}, \qquad (17)$$

where $\lambda > 0$ does not depend on *n*, and the following relations is satisfied between the difference derivation $\frac{g_n(x_{0k})-g_n(x_{0k-1})}{h}$ and $g'(x_k)$:

$$\left| g'(x_k) - \frac{g_n(x_{0k}) - g_n(x_{0k-1})}{h} \right| \le \Theta(x_k), \quad k = \overline{2,n}, \tag{18}$$

where the value $\Theta(x_k), \quad k = \overline{2,n}$ satisfies the relations (10), (11).

Proof. Using the representation (12), we write the system (16) in the equivalent form:

$$\sum_{k=1}^{n+1} \frac{g(x_{0k}) - g_n(x_{0k-1})}{h} Li\left(\frac{c}{x_{0j} - x_k}\right) = f(x_{0j}), \quad j = \overline{1,n}.$$

$$\sum_{k=1}^{n+1} \frac{g(x_{0k}) - g_n(x_{0k-1})}{h} h = 0, \quad j = 0,$$

it is believed in formulas that $g(x_{00}) = g(x_{0n+1}) = 0.$

Using the numerical solution to the singular integral equation via the discrete vortex method, for the index *κ = 1* at the value of the integral of the solution equal to zero, we have:

$$\frac{g(x_{0k}) - g_n(x_{0k-1})}{h} = \frac{1}{h} I_{1,k}^{(n+1)} \sum_{j=1}^{n} \frac{1}{h} I_{1,0j}^{(n+1)} \frac{f(x_{0j})}{x_k - x_{0j}}, \quad k = \overline{1,n+1},$$

where $I_{1,k}^{(n+1)}, I_{1,0j}^{(n+1)}$ are specified in the study [5], therefore, according to [5], the relations (18), (17) are correct.

Summary

The results of the article can be used to solve the problems that can be reduced to integral equations with fractional order in simulating a wide class of the problems that are applied in various fields, especially in simulating some fractional (or fractal) dynamic processes.

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