DOMATIC NUMBERS OF SOME PARTICULAR CLASSES OF ECCENTRIC GRAPHS
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Abstract
Along with the growth of several research areas in graph theory, the study of domination in graphs has witnessed a speedy growth in the past few decades. The concept of domination is widely used in the areas which are mainly associated with the location of service centers. In social networks it is used in the selection of representatives. Its application plays a vital role in various areas such as social network problems, biological model networks, service locations problems, communication networks, etc. The speedy growth and the increasing nature of applications of the concept of domination have caused the extension of the notion ‘domination numbers’ to another notion, ‘domatic numbers’ in graphs. Having motivated by the huge uprising and widely applying nature of the theory of domination in graphs, in this article, we study the parameter of domatic numbers of a special class of graphs namely, ‘ Eccentric graphs’ for some special cases.

Keywords: vertices, edges, graphs, connected graphs, induced subgraphs, domination, domatic numbers,

1. Introduction
The growth of graph theory is almost inevitable almost everywhere in various fields of science and technology from the past century. Graph theoretic techniques play a vital role for the analysis of communication networks. The relationship between various objects with some common feature can be modelled mathematically as a graph where the objects are represented by nodes or vertices and the joins or edges between the vertices represent the relationship between them.

The notion of domination was formally formulated by the mathematician Ore in [1]. He defined a dominating set in any graph to be the set of vertices $D$ such that the remaining vertices are adjacent to at least any one of the vertices of $D$. The least size of dominating sets of a graph $G$ is said to be its domination number and is denoted as $\gamma(G)$. 
Claude Berge [2], dealt with the concept of domination for the first time in his book, followed by the works of Oystein Ore [1], who used the terms ‘dominating set’ and ‘domination number’ firstly. In 1977, Cockayne and Hedetniemi surveyed few results available about domination. Since its inception, the concept of domination has found rapid growth due to overwhelming contributions of researchers and others towards the theory. Any set of vertices $D$, besides being a dominating set, can further be identified as a connected [3] or independent [4] or total (open) [5] dominating set according as the subgraph $\langle D \rangle$ induced by $D$ is connected or null graph or without isolates. The domination number for any domination parameter is the cardinality of the corresponding minimum dominating set. The connected domination number, independent domination number and total domination numbers are respectively denoted by the symbols $\gamma_c(G), \gamma_i(G), \gamma_t(G)$. Furthermore, not all graphs possess all types of domination parameters. For example, there is no connected dominating set for a disconnected graph.

Clearly, in any graph $G$, $\gamma_c(G), \gamma_i(G), \gamma_t(G) \geq \gamma(G)$.

Besides these, in the theory of domination, there is another notion namely, ‘Domatic partition’ or $D$-partition of graphs introduced by Cockayne and Hedetniemi in [6].

A $D$-partition or domatic partition in a graph $G$, is a partition of $V(G)$ into dominating sets. The maximum number of classes of a $D$-partition is called as the domatic number of $G$ denoted by $d(G)$. A graph $G$ is called domatically full if $d(G) = \delta(G) + 1$, where $\delta(G)$ is the minimum degree of $G$. The connected (or independent or total) domatic partition is a partition of $V(G)$ into connected (or independent or total) dominating sets. The numbers of such domatic parameters are the maximum order of the corresponding domatic partitions of $V(G)$. Similar to domination parameters, the symbols $d_c(G), d_i(G), d_t(G)$ denote the connected, independent, total domatic numbers of the graph $G$. The notion of domatic numbers in graphs was introduced by Cockayne and Hedetniemi in [6]. The same authors and Dawes introduced the notion of total domatic numbers in [7]. The connected domatic numbers were introduced by Laskar and Hedetniemi in [8]. The independent domatic number is also known as idomatic number of a graph. The idomatic number of a graph is studied by Zelinka in [9]. As in [6], we let $d_i(G) = 0$, if there is no independent domatic partition of $V(G)$ and we mean similarly with other domatic parameters also. Any interested reader can go through the book written by Haynes et al., [10] or any other related sources to know more on the growth and development of the concept of domination.
domination in graphs. Turning to eccentric graphs, eccentricity associated characterizations of graphs also sound good in literature. Akiyama et al [11] and Buckley [12] have respectively discussed self-centered or equi-eccentric graphs. Unique eccentric point graphs were analyzed by Parthasarathy and Nandakumar in [13]. Several such studies focusing at the variant of eccentricity can be found from the references of these research articles cited therein. **Eccentric graphs** are one of those classes of graphs resulting out of the notion of eccentricity in graphs, contributed by Akiyama et al in [14]. They have also identified few of its interesting properties.

On attaining enough motivations on the notions of domatic numbers and eccentric graphs we study the variant *domatic numbers* for some particular classes of eccentric graphs. The survey about characterizations of eccentric graphs for standard classes of graphs depicts marvelous graph structures and therefore the matter of study would yield wonderful results and properties. Unless and otherwise stated, the terms and notions in this article are according to Harary [15].

### 2. Eccentric Graphs

Terms related with eccentricity will be often used in this article. A vertex *u* is an *eccentric vertex* of another vertex *v* if *u* is a farthest vertex from *v* and the set of all eccentric vertices of *v* are denoted as *E(v)*. If *G* is a disconnected graph, 

\[ d(u, v) = \infty, \]

where the vertices *u, v* are from different components. Thus, in case of disconnected graphs, vertices belonging to different components will always be eccentric vertices to each other.

#### 2.1. Definition [14]

A graph \( G_e \) is said to be an *eccentric graph* of another graph *G*, if they are defined on the same set of vertices and any two vertices of *G_e* are adjacent if and only if at least any one of them is an eccentric vertex of the other.

Obviously, **eccentric graphs** are connected graphs, as there exist one or more eccentric vertices for every vertex of the graph. Jin Akiyama et al. introduced the notion of eccentric graphs in [14] together with the contributions of few of its interesting characterizations and properties on certain classes of graphs.

### 3. Main Results

In this section we discuss the domatic numbers of eccentric graphs of some special classes of graphs.

In a graph *G*, any vertex adjacent to a pendant vertex of *G* is said to be a support of *G*. We let \( S_s \) denote the set of pendant vertices adjacent to a support vertex *u* is denoted as \( S_u \). We recall that a double star is often denoted by \( S_{a,b} \). A graph with \( p \) disjoint copies of \( K_2 \) is denoted as \( pK_2 \).
The eccentric graphs were characterized by Gayathri and Kaspar in [16]. In order to make the article self-contented, we revise some of the important results from [16] wherever required.

I. Paths

3.1. Proposition [16]

Let \( G \) be a path of order \( n \). Then,

(i) \( G_e \cong K_n \), if \( n \leq 3 \),

(ii) If \( n > 3 \) is even, \( G_e \cong S_{k,k} \) with \( k = \frac{n-2}{2} \).

(iii) If \( n > 3 \) is odd, \( G_e \cong H \) where \( H \) is a triangle whose any two vertices are attached with \( \frac{n-3}{2} \) number of pendant edges each.

3.2. Theorem

If \( G \cong P_n \), then

(i) \( d(G_e) = \begin{cases} n, & n \leq 3 \\ 2, & n > 3 \end{cases} \)

Moreover, the graph \( G_e \) is domatically full.

(ii) \( d_i(G_e) = \begin{cases} n, & n \leq 3 \\ 2, & n > 3 \text{ and is even} \\ 0, & n > 3 \text{ and is odd} \end{cases} \)

(iii) \( d_e(G_e) = d_i(G_e) = 1 \)

Proof

(i) If \( n \leq 3 \), then \( G_e \cong K_n \) and therefore each vertex is a dominating set and hence the result follows. Also complete graphs are domatically full graphs.

Let \( n > 3 \). We note that the two support vertices say \( u \) and \( v \) of the graph \( G_e \) is a \( \gamma \)-set. Also, the set \( V \setminus \{u,v\} \) is another dominating set for \( G_e \). Clearly, these two dominating sets form a partition for vertex set \( V \) and one can easily verify that there exists no more \( D \)-partitions with more dominating classes. Therefore, \( d(G_e) = 2 \). Since \( \delta(G) = 1 \), it easily follows that \( G_e \) is domatically full.
(ii) Assume that \( n \leq 3 \). Then \( G_e \cong K_n \) and the result is immediate.

Let \( n > 3 \) be even. This implies by proposition 3.1, \( G_e \cong S_{k,k} \), a double star. Let \( u \) and \( v \) be the two centers of \( S_{k,k} \).

Then \( S_1 = \{u, S_v\} \), \( S_2 = \{v, S_u\} \) are the two \( \gamma_i \)-sets of \( G_e \) whose union is \( V(G) \). The order of this partition is maximum is clear from the proposition 3.1. Hence \( d_i(G_e) = 2 \). If \( n > 3 \) is odd, then \( S_1 \cup S_2 = V(G) - \{w\} \) where the vertex \( w \) is adjacent to both vertices \( u \) and \( v \). Therefore, no such partition exists in this case implying that \( d_i(G_e) = 0 \).

(iii) From proposition 3.1, we perceive that \( D = V(G) \) is the only partition of \( V(G) \) for connected and total domatic partitions and therefore \( d_c(G_e) = d_i(G_e) = 1 \).

II. Cycles

3.3 Proposition [16]

Let \( G \) be a cycle of order \( n \geq 3 \). Then

(i) \( G_e \cong mK_2 \), if \( n \) is even where \( m = n/2 \).

(ii) \( G_e \cong C_n \), if \( n \) is odd.

3.4 Theorem

If \( G \cong C_n \), then

\[
d(G_e) = \begin{cases} 3, & \text{if } n = 3(2k-1), k = 1, 2, \ldots, \\ 2, & \text{otherwise} \end{cases}
\]

Moreover, the graph \( G_e \) is domatically full when \( n = 3(2k-1), k = 1, 2, 3, \ldots \)

(ii) \( d_c(G_e) = \begin{cases} n, & \text{if } n \leq 3 \\ 2, & \text{n > 3 and is even} \\ 0, & \text{n > 3 and is odd} \end{cases} \)

(iii) \( d_c(G_e) = d_i(G_e) = 1 \)

Proof

Assume \( n \) to be odd. Then from proposition 3.3, we have, \( G_e \cong C_n \).

From [5],

\[
d(C_n) = \begin{cases} 3, & \text{if } n = 3k, k = 1, 2, \ldots \\ 2, & \text{otherwise} \end{cases}
\]

Rewriting \( n = 3k \) for odd terms only we get, \( n = 3(2k-1), k = 1, 2, \ldots \) and thus, \( d(C_{3(2k-1)}) = 3, k = 1, 2, \ldots \) Since, \( \delta(C_n) = 2 \), we find that \( C_n \) is domatically full in this case. Since \( d(C_n) = 2 \), if \( n \neq 3k \) (see [5]), we get \( d(C_n) = 2 \), when \( n \) is odd but \( n \neq 3(2k-1), k = 1, 2, \ldots \)
The proof will be complete if we show that $d(C_n) = 2$, when $n$ is even also.

Assume $n$ to be even. Then $G_e \cong mK_2$, with $m = n/2$, by proposition 3.3. Let $S_1, S_2$ be two different subsets of $V$, for every edge in $G_e$ its one end vertex is in $S_1$ and the other in $S_2$. Clearly, $S_1$ and $S_2$ are $\gamma$-sets forming a $D$-partition for $V(G_e)$ implying that $d(G_e) = 2$.

III. Complete Graphs

3.5. Proposition [14]

$G = G_e = K_n$ for any complete graph $K_n$ of order $n$.

3.6. Theorem

If $G \cong K_n$, then

(i) $d(G_e) = d_e(G_e) = d_i(G_e) = n$ and the graph $G_e$ is domatically full.

(ii) $d_i(G_e) = \lfloor n/2 \rfloor$.

We remark that if $G$ is any graph such that for all $v \in V(G)$, either $e(v) = 1$ or $e(N(v)) = 1$ (i.e. every neighbour of $v$ has eccentricity 1) then $G_e = K_n$ and hence the theorem 3.6 holds good.

IV. Complete Bipartite Graphs

3.7. Proposition [16]

Let $G \cong K_{m,n}$ be a complete bipartite graph with $m \leq n$. Then,

(i) if $m = 1$, $G_e \cong K_{n+1}$,

(ii) if $m > 1$, $G_e \cong K_m \cup K_n$, an union of two vertex-disjoint complete subgraphs.

3.8. Theorem

If $G \cong K_{m,n}$, with $m \leq n$, then

(i) $d(G_e) = \begin{cases} n+1, & \text{if } m = 1 \\ m, & \text{if } m > 1 \end{cases}$.

Furthermore, the graph $G_e$ is domatically full.

(ii) $d_i(G_e) = \begin{cases} n+1, & \text{if } m = 1 \\ 0, & \text{otherwise} \end{cases}$
(iii) \[ d_i(G_e) = \begin{cases} n+1, & \text{if } m = 1 \\ m, & \text{if } 1 < m = n \\ 0, & \text{otherwise} \end{cases} \]

(iv) \[ d_i(G_e) = \begin{cases} \lfloor (n+1)/2 \rfloor, & \text{if } m = 1 \\ \lceil (m+1)/2 \rceil, & \text{otherwise} \end{cases} \]

Proof

Let \( G \cong K_{m,n} \), with \( m \leq n \).

(i) If \( m = 1 \), then \( G_e \cong K_{n+1} \), by proposition 3.7 and therefore each vertex is a \( \gamma \)-set resulting to \( d(G_e) = n + 1 \).

If \( m > 1 \), then \( G_e \cong K_m \cup K_n \), a vertex-disjoint union of complete subgraphs.

Set \( S_i = \{u_i,v_i\}, i = 1, 2, \ldots, m - 1 \) such that
\[ u_i \in V(K_m), v_i \in V(K_n) \text{ and } S_i \cap S_j = \phi, i \neq j. \]

Set \( S_m = V(G_e) - \bigcup_{i=1}^{m-1} S_i \).

Clearly, the vertex subsets \( S_i \) are well defined and \( \bigcup_{i=1}^{m} S_i = V(G_e) \), \( S_i \cap S_j = \phi, i \neq j \) from which it follows that the subsets \( S_i \) form a vertex partition for \( G_e \) and each is a dominating set for \( G_e \). Since \( m \leq n \), no more such \( D \)-partition exists with higher order. Therefore, \( d(G_e) = m \) and the fact that \( G_e \) is domatically full is easily evident from which the proof for (i) is complete.

(ii) If \( m = 1 \), then \( G_e \cong K_{n+1} \) and therefore, \( d_e(G_e) = n + 1 \); otherwise \( G_e \) is disconnected and hence no connected dominating set exists for \( G_e \) as a result of which \( d_e(G_e) = 0 \).

(iii) Since each component of \( G_e \) is complete, every independent dominating set of \( G_e \) consists exactly one vertex from each component. Assume \( m = 1 \). Then \( G_e \cong K_{n+1} \) and therefore, \( d_1(G_e) = n + 1 \) is easily understood. Assume \( m > 1 \). If \( m = n \), each \( S_i \) is defined above is an independent dominating set and the \( D \)-partition discussed becomes an independent \( D \)-partition implying the result that \( d_i(G_e) = m \). If \( m < n \), then any \( D \)-partition of \( V(G_e) \) includes at least one set \( S_i \) consisting of two or more vertices from the same component of \( G_e \) so that \( S_i \) is no longer an independent
dominating set for $G_e$ and consequently no independent $D$-partition exists resulting to $d_i(G_e) = 0$. Hence (iii) is proved.

(iv) As $G_e$ is a complete graph with $n+1$ vertices if $m=1$, any two vertices form a total dominating set for $G_e$ in this case and therefore $d_i(G_e) = \left\lfloor (n+1)/2 \right\rfloor$. When $m > 1$, $G_e$ is disconnected whose two components are complete subgraphs of orders $m$ and $n$. Therefore, any two vertices from each component of $G_e$ form a total dominating set for $G_e$. Since $m \leq n$, we observe that $\left\lfloor (m+1)/2 \right\rfloor$ such disjoint total dominating sets exist in $G_e$ reducing the fact that $d_i(G_e) = \left\lfloor (m+1)/2 \right\rfloor$ which is statement (iv) completing the proof of the theorem.

V. Hypercubes

3.9 Proposition [16]

If $G \cong Q_n$ be an $n$-dimensional hypercube, then $G_e \cong mK_2$, $m = 2^{n-1}$.

3.10 Theorem

If $G \cong Q_n$, is a hypercube, then

(i) $d(G_e) = d_i(G_e) = 2$. Moreover, the graph $G_e$ is domatically full.

(ii) $d_e(G_e) = \begin{cases} 2, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$

(iii) $d_i(G_e) = 1$.

Proof

Let $G \cong Q_n$ be a hypercube. This implies by theorem 3.9, $G_e \cong mK_2$, with $m = 2^{n-1}$.

Let $S_1$ and $S_2$ be a partition of vertices of $G_e$ such that the for each $K_2$ of $G_e$ one end vertex is in $S_1$ and the other in $S_2$. Clearly, $S_1$ and $S_2$ are both $\gamma$-sets and $\gamma_i$-sets for $G_e$. Consequently, $d(G_e) = d_i(G_e) = 2$ and since $\delta(G_e) = 1$, $G_e$ is domatically full. Thus we get, (i).

If $n = 1$, then $G_e = K_2$; otherwise $G_e$ is disconnected and hence we get (ii).

Since, $\gamma_i(G_e) = n$ by inspection, $d_i(G_e) = 1$ which is the assertion (iii). This completes the proof of the theorem.
VI. Wheels

3.11 Proposition [16]

Let \( G \cong W_n \) be a wheel of order \( n+1 \). Then,

(i) \( G \cong G_e \cong W_n \cong K_4 \), if \( n = 3 \)

(ii) \( G_e \cong C_n + K_1 \), if \( n > 3 \)

3.12 Theorem

If \( G \cong W_n \) be a wheel on \( n+1 \) vertices, then

(i) \( d(G_e) = \begin{cases} 4, & \text{if } n = 3 \\ \left\lfloor n/2 \right\rfloor + 1, & \text{if } n > 3 \end{cases} \)

(ii) \( d_i(G_e) = \begin{cases} n/2 + 1, & \text{if } n > 3 \text{ & is even} \\ 0, & \text{if } n > 3 \text{ & is odd} \end{cases} \)

(iii) \( d_c(G_e) = \begin{cases} 4, & \text{if } n = 3 \\ 1, & \text{if } n = 4 \\ 2, & \text{if } n = 5 \\ \left\lfloor n/2 \right\rfloor + 1, & \text{if } n \geq 6 \end{cases} \)

(iv) \( d_i(G_e) = \begin{cases} 2, & \text{if } n = 3,5 \\ \left\lfloor n+1 \right\rfloor, & \text{if } n \geq 6 \end{cases} \)

Proof

Let \( G \cong W_n \) be a wheel on \( n+1 \) vertices which is given by \( C_n + K_1 \)

To be precise and concise, we will prove the result for various domination parameters parallelly for different values of \( n \).

By theorem 3.11, \( G_e \cong K_4 \), if \( n = 3 \) and \( G_e \cong C_n + K_1 \), if \( n > 3 \).

Let \( u \) be the central vertex of the wheel \( W_n \).
Let \( n = 3 \), then \( G_e \cong K_4 \) and each vertex of \( G_e \) is a \( \gamma \)-set and \( \gamma_c \) and \( \gamma_i \)-sets as well. Therefore, 
\[
d(G) = d_i(G) = d_c(G) = 4.
\]
Further, any two vertices of the graph \( G_e \) form a \( \gamma_i \)-set and hence \( d_i(G) = 2 \).

Let \( n > 3 \). Clearly, \( \{u\} \) is the minimum dominating set for \( G_e \). We observe that, for the remaining vertices \( v \in V(C_n) \), 
\[
N_{G_n}(v) = V(G) - N[v].
\]
Therefore, any two adjacent vertices of \( C_n \) form a dominating set for \( G_e \). Thus there are \( \left\lfloor \frac{n}{2} \right\rfloor \) such disjoint dominating sets which form a partition for the vertices of the cycle. So, on the whole, there are \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) such dominating sets which partition \( V(G_e) \) implying the result that \( d(G_e) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

Further, if \( n \) is even we note that such dominating sets are independent, since in this case any two adjacent vertices of \( G \) are independent (nonadjacent) in \( G_e \). Hence, \( d_i(G) = n/2 + 1 \). If \( n \) is odd, then any domatic partition must include, at least one dominating set with 3 or more vertices which cannot be independent since any vertex in \( G_e \) is nonadjacent to at most 2 vertices (that are two adjacent vertices of the cycle in \( G \) ). Hence no such independent domatic partition exists and thereby we get, \( d_i(G) = 0 \) if \( n \) is even.

The proof concerning assertions (i) and (ii) is complete. The remaining part is the proof for assertions (iii) and (iv) when \( n > 3 \).

Assume \( n > 3 \). We recall that \( G_e \cong \overline{C_n} + K_1 \), if \( n > 3 \).

In particular, let \( n = 4 \). Then \( G_e \cong 2K_2 + K_1 \). In this case every connected dominating set includes the central vertex \( u \) and therefore, only one connected domatic partition can exist for the graph \( G_e \) and hence \( d_c(G) = 1 \) and by the same token, \( d_i(G) = 1 \).

Let \( n = 5 \). Then, \( G \cong G_e \cong W_5 \).

In this case any three consecutive vertices of the cycle \( C_5 \) and the remaining vertices of \( W_5 \) form a maximum connected domatic partition for \( G_e \) which is easy to observe and hence \( d_c(G) = 2 \). Similarly, one can easily verify that \( d_i(G) = 2 \).

Let \( n \geq 6 \). Then, \( G_e \cong \overline{C_n} + K_1 \).

From the graph \( G_e \) we infer the following. (a) \( \{u\} \) is the minimum dominating set for \( G_e \) and any other dominating set contains at least two vertices. (b) Though any two adjacent vertices of the cycle \( C_n \) is a dominating set they are not
connected in $G$. (c) any two alternate vertices of the cycle $C_n$ are not dominating sets, since, their common vertex in $G$ is not adjacent to any of them in $G$. (d) Let $C_n = v_1v_2 \ldots v_nv_1$ be the biggest cycle in $G$. Then $S_i = \{v_i, v_{i+3}\}$, $1 \leq i \leq \left\lfloor n/2 \right\rfloor$ is a smallest connected dominating set other than $\{u\}$. Let $S = V(G) - \bigcup_{i=1}^{k} S_i$ where $k = \left\lfloor n/2 \right\rfloor$. This implies $u \in S$ and hence $S$ is also a connected dominating set. Thus we see that the vertex sets $S_i$'s form a connected domatic partition of maximum order. Therefore, $d_c(G_e) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

We realize that the similar arguments hold good for total domatic partition also when the classes of dominating sets in the connected domatic partition are of size two. Since, every total dominating set has at least two vertices, the central vertex $\{u\}$ of $G_e$ must also be included in any of the above dominating sets $S_i$. Thus we get, $d_t(G_e) = \left\lfloor \frac{n+1}{2} \right\rfloor$.

VII. Fans

3.13. Proposition [16]

Let $G \cong F_n$, $n \geq 2$ be a fan graph of order $n+1$. Then,

(i) $G_e \cong K_{n+1}$, if $n \leq 3$

(ii) $G_e \cong P_n + K_1$, if $n > 3$

3.14. Theorem

If $G \cong F_n$, $n \geq 2$ be a path on $n+1$ vertices, then

(i) $d(G_e) = \begin{cases} n+1, & \text{if } n \leq 3 \\ \left\lfloor n/2 \right\rfloor + 1, & \text{if } n > 3 \end{cases}$

(ii) $d_t(G_e) = \begin{cases} n+1, & \text{if } n \leq 3 \\ \frac{n}{2} + 1, & \text{if } n > 3 \text{ & is even} \\ 0, & \text{if } n > 3 \text{ & is odd} \end{cases}$

(iii) $d_c(G_e) = \begin{cases} n+1, & \text{if } n \leq 3 \\ 2, & \text{if } n = 4 \\ \left\lfloor n/2 \right\rfloor + 1, & \text{if } n \geq 5 \end{cases}$
(iv) \[ d_r(G_e) = \left\lfloor \frac{n+1}{2} \right\rfloor. \]

**Proof**

Let \( G \cong F_n, \ n \geq 2 \) be a fan graph on \( n+1 \) vertices. Then \( G \cong F_n \cong P_n + K_1 \).

When \( n \leq 3 \), \( G_e \cong K_{n+1} \). Therefore the result that \( d(G) = d_r(G) = d_c(G) = n+1 \) is trivial.

Assume \( n > 3 \). Let \( V(K_1) = \{u\} \). As in the case of wheels any two adjacent vertices of \( P_n \) form a dominating set apart from the dominating set \( \{u\} \). Therefore \( d(G_e) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \). As in the proof of the theorem on wheels, the assertion (ii) is easy to realize and we omit the same.

Let the path \( P_n \) of the fan \( F_n \) be \( P_n = v_1v_2 \ldots v_n \).

When \( n = 4 \), it is easy to observe that \( \{v_1, v_4\} \) and \( \{u, v_2, v_3\} \) is a connected domatic partition of maximum order and hence \( d_c(G_e) = 2 \).

When \( n > 4 \), we understand that \( S_i = \{v_i, v_{i+3}\}, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \) is always a connected dominating set and there is no more minimum dominating sets other than \( \{u\} \). Also, Let \( S = V(G) - \bigcup_{i=1}^{k} S_i \) where \( k = \left\lfloor \frac{n}{2} \right\rfloor \) is also a connected dominating set. Clearly, the vertex subsets \( S \) and \( S_i \)'s form a partition for \( V(G_e) \). Hence \( d_c(G_e) = \left\lfloor \frac{n}{2} \right\rfloor + 1, \ if \ n \geq 5 \).

When \( n \leq 3 \), \( G_e \cong K_{n+1} \). Therefore, \( d(G_e) = \left\lfloor \frac{n+1}{2} \right\rfloor \).

When \( n > 3 \), each dominating set in the connected domatic partition is also a total dominating set except the dominating set \( \{u\} \) which has to be included in any other dominating set of the partition so that the connected domatic partition is also total domatic partition. Therefore, we obtain, \( d(G_e) = \left\lfloor \frac{n+1}{2} \right\rfloor \) which concludes the proof of the theorem.

**VIII. Stars**

From [16], the eccentric graph of a star \( K_{1,n} \) is a complete graph \( K_{n+1} \). Therefore, the domatic parameters of the eccentric graphs of stars are same as those of the complete graphs \( K_{n+1} \) and evidently the theorem 3.6 (which would be monotonic to repeat) hold good in this case.
IX. Double Stars

3.15. Proposition [16]

If \( G \cong S_{m,n} \), is a double star graph of order \( m+n+2 \), then \( G_e \cong K_1 + \overline{K_m} + \overline{K_n} + K_1 \).

3.16. Theorem

If \( G \cong S_{m,n} \), \( m \leq n \), be a double star of order \( m+n+2 \), then

(i) \( d(G_e) = m+1 \)
(ii) \( d_e(G_e) = d_i(G_e) = m \)
(iii) \( d_i(G_e) = 2 \)

Proof

Let \( G \cong S_{m,n} \), \( m \leq n \) be a double star graph having \( m+n+2 \) vertices. Since, by symmetry we have \( S_{m,n} \cong S_{n,m} \), it is easy to consider any double star \( S_{m,n} \) such that \( m \leq n \).

Let, \( u,v \) be the two centers of the double star which themselves make up a dominating set for \( G_e \). The eccentric graph \( G_e \) of the double star \( S_{m,n} \) contains the complete bipartite subgraph \( K_{m,n} \) with \( m \leq n \). Let \( M \) be a maximum matching of \( K_{m,n} \subset G_e \). Clearly, \( |M| = m \), as \( m \leq n \). Let \( S_i \) be the end vertices of each edge \( e_i \), \( i = 1,2, \ldots m \) in \( M \). Then, each \( S_i \) is a dominating set for \( G_e \). Let \( S = V(G_e) - \bigcup_{i=1}^{m} S_i \). Since \( u,v \in S \), \( S \) is also a dominating set for \( G_e \). Evidently, the vertex subsets \( S \) and \( S_i \)'s form a domatic partition of maximum order for \( V(G_e) \) and therefore, \( d(G_e) = m+1 \).

Let \( S_i \) and \( S \) be as defined above. Then, each \( S_i \) defined above is a minimum connected as well as total dominating set. Hence \( d_e(G_e) = d_i(G_e) \geq m \). Although the set \( S \) is a dominating set, we find that its induced subgraph \( \langle S \rangle \) is disconnected. We modify the sets \( S_i \) by adding the vertices of \( S \) to them arbitrarily. We realize that any such sharing does not affect the connectivity of the dominating sets \( S_i \). Hence the sets \( S_i \), \( i = 1,2, \ldots m \) form a connected domatic partition of maximum order for \( V(G_e) \). Therefore, \( d_e(G_e) = d_i(G_e) \leq m \) by which assertion (ii) is clear.

Clearly, the subgraph \( K_{m,n} \subset G_e \) has an independent vertex partition of maximum order 2, say \( V_1 \) and \( V_2 \) but yet they are not dominating sets for \( G_e \). Clearly, each center is independent to exactly any one the sets of this partition say \( u \) is independent with \( V_1 \) and \( v \) with \( V_2 \). Then the sets \( V_1 \cup \{u\} \) and \( V_2 \cup \{v\} \) are dominating sets forming an independent
domatic partition of maximum order 2 for $V(G_e)$. Therefore, $d_i(G_e) = 2$ which is the assertion (ii). This ends the proof of the theorem.

X. Disconnected Graph

3.17 Proposition [16]

Let $G$ be a disconnected graph with two components $H_1$ and $H_2$ such that $|V(H_1)| = m$ and $|V(H_2)| = n$. Then, $G_e \cong K_{m,n}$.

3.18 Theorem

If $G$ is a disconnected graph having two components, $H_1$ and $H_2$ such that $|V(H_1)| = m$, $|V(H_2)| = n$ and $m \leq n$, then

(i) $d(G_e) = d_e(G_e) = d_i(G_e) = m + n$

(ii) $d_i(G_e) = m$

Proof

Let $G$ be a disconnected graph as stated in the theorem. Then by proposition 3.17, $G_e \cong K_{m,n}$. Since each vertex in $G_e$ is a minimum dominating set, result (i) is easily evident.

Let $M$ be a maximum matching in $G_e \cong K_{m,n}$. The matching number for the graph $G_e$ is $\beta_i(G_e) = m$. Let $S_i, i = 1, 2, \ldots, m$ be the set of end vertices of any edge $e_i \in M$. We note that the sets $S_i$ form a total dominating set for the eccentric graph $G_e$. The remaining vertices (not in any subset $S_i$) if exist are independent and hence cannot form any total dominating set but, arbitrary addition of such vertices to sets $S_i$ does not affect their total dominance property. Hence, $d_i(G_e) = m$ which completes the proof of the theorem.

4. Conclusion and Future Scope

We have made enough attempt to characterize various domatic numbers for various classes of eccentric graphs in this article. For this purpose, we have considered four types domatic parameters, viz., domatic number, connected domatic number, independent domatic number and total domatic number for scrutiny. As much as ten special classes of graphs such as paths, cycles, complete graphs, etc were investigated in this article.
As a future work, the readers may extend the results to more variety of classes of graph. Results may be generalized and related with other graph theoretical parameters like, size, order, degree, independence numbers, etc.

References


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